From Artin monoids to Artin groups

Joint with Rachael Boyd & Rose Morris-Wright

Notation

Γ = graph with vertices $\{s_1, \dots, s_n\}$ and edges labelled by $\overset{m_{ij}}{\underset{s_i \quad s_j}{\bullet\!\!-\!\!-\!\!\bullet}}$, $m_{ij} \in \{2, 3, 4, \dots\}$

$$A_\Gamma = \langle s_1, \dots, s_n \mid \underbrace{s_i s_j s_i \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}} \rangle$$

A_Γ^+ = monoid generated by same presentation

W_Γ = associated Coxeter

Two classes of Artin groups:

A_Γ spherical type $\Leftrightarrow$ W_Γ finite

A_Γ infinite type $\Leftrightarrow$ W_Γ infinite

For spherical type Artin groups, every element can be written in the form $a\Delta^{-n}$, $a \in A_\Gamma^+$, Δ = Garside element.

$\Downarrow$

Many questions about A_Γ can be reduced to questions about A_Γ^+.

For infinite type, A_Γ^+ is still has many nice properties, but passing from A_Γ^+ to A_Γ is much more difficult!

What do we know about A_Γ^+?

For any Γ,

- There is a well-defined length function $\ell : A_\Gamma^+ \to \mathbb{Z}$ by $s_i \mapsto 1$

- The word problem in A_Γ^+ is solvable.

- (Brieskorn – Saito, 1972) There is a partial ordering on A_Γ^+ satisfying nice properties
  Eg: $a \leq_L b$ if $\exists c$ such that $ac = b$
  If $X \subset A_\Gamma^+$ is a finite set, it has a ! $\gcd_L(X)$

- (J. Michel, 1999) There is a nice normal form for elts of A_Γ^+ with respect to the generating set $\{\Delta_T \mid A_T \subseteq A_\Gamma \text{ spherical type}\}$
  Garside elt

A_Γ^+ has nice combinatorial structure & is much easier to understand than A_Γ for infinite type Artin groups.

What is the relation between A_Γ^+ and A_Γ?

- (Paris, 2002) The natural map $A_\Gamma^+ \to A_\Gamma$ is injective for all A_Γ.

- Let $\mathcal{H}_\Gamma$ = hyperplane complement associated to $W_\Gamma \curvearrowright \mathbb{C}^n$

Then $\pi_1(\mathcal{H}_\Gamma / W_\Gamma) = A_\Gamma$ (van der Lek, 1983)

Deligne: If W_Γ is finite (i.e. A_Γ is spherical type), then $\mathcal{H}_\Gamma / W_\Gamma$ is a $K(\pi,1)$-space for A_Γ, that is,

$$\mathcal{H}_\Gamma / W_\Gamma \simeq BA_\Gamma$$

$$\boxed{K(\pi,1)\text{-conjecture: } \mathcal{H}_\Gamma / W_\Gamma \simeq BA_\Gamma \text{ for all } A_\Gamma.}$$

Monoids also have classifying spaces, and

$$\pi_1(BA_\Gamma^+) \cong \pi_1(BA_\Gamma) = A_\Gamma$$

<u>Thm</u> (Dobrinskaya 2006, Ozornova, Paolini 2017)

For all A_Γ, $\mathcal{H}_\Gamma / W_\Gamma \simeq BA_\Gamma^+$

$$\boxed{\text{so } K(\pi,1)\text{-conj holds iff } BA_\Gamma^+ \to BA_\Gamma \text{ is a homotopy equivalence.}}$$

Geometric relations between A_Γ^+ and A_Γ

I. Deligne complex, $\mathcal{D}_\Gamma$

Joint work with R. Boyd & R. Morris-Wright

$\mathcal{D}_\Gamma$ is the cubical complex with

vertices: aA_T, $a \in A_\Gamma$, $A_T \subseteq A_\Gamma$ spherical type

edges: $aA_T \longrightarrow bA_R$, $aA_T \subseteq bA_R$, $|R \setminus T| = 1$

k-cube: $[aA_T, bA_R]$, $aA_T \subseteq bA_R$, $|R \setminus T| = k$

Thm (C-Davis, 1995) $\mathcal{D}_\Gamma \simeq \widetilde{\mathcal{H}_\Gamma / W_\Gamma}$ univ cover.

Thus $K(\pi,1)$-conj holds $\iff$ $\mathcal{D}_\Gamma$ is contractible.

In particular, this holds if $\mathcal{D}_\Gamma$ has a CAT(0) metric, (eg. if A_Γ is FC-type, 2-dim'l, locally reducible, ...)

Define a Deligne complex $\mathcal{D}_\Gamma^+$ for A_Γ^+

vertices: cosets of A_T^+, A_T spherical type

$a \in A_\Gamma^+$, $aA_T \cap A_\Gamma^+ = \bar{a} A_T^+$ $\quad$ $\bar{a}$ = min length elt

k-cubes: $[\bar{a} A_T^+, \bar{b} A_R^+]$, $\bar{a} A_T^+ \subseteq \bar{b} A_R^+$, $|R \setminus T| = k$

There is a natural map $\mathcal{D}_\Gamma^+ \longrightarrow \mathcal{D}_\Gamma$ by $\bar{a} A_T^+ \mapsto \bar{a} A_T$

Thm (B-C-MW) For any Artin group A_Γ

① The natural map $\mathcal{D}_\Gamma^+ \to \mathcal{D}_\Gamma$ is an embedding

② With respect to the cubical metrics on $\mathcal{D}_\Gamma^+$, $\mathcal{D}_\Gamma$, this embedding is locally isometric.

Cor If A_Γ is FC-type, then $\mathcal{D}_\Gamma^+ \hookrightarrow \mathcal{D}_\Gamma$ is (globally) isometric and $\mathcal{D}_\Gamma^+$ is CAT(0).

Question: Is $\mathcal{D}_\Gamma^+ \to \mathcal{D}_\Gamma$ locally isometric with respect to the Moussong metric?

Thm (B-C-MW) For any Artin group A_Γ, $\mathcal{D}_\Gamma^+$ is contractible.

Pf: Uses the combinatorial structure of A_Γ^+.

Question: We can cover $\mathcal{D}_\Gamma$ with translates $\mathcal{D}_\Gamma^+$ and study the intersections of these translates. Are there conditions under which one can show that $\mathcal{D}_\Gamma$ is also contractible?

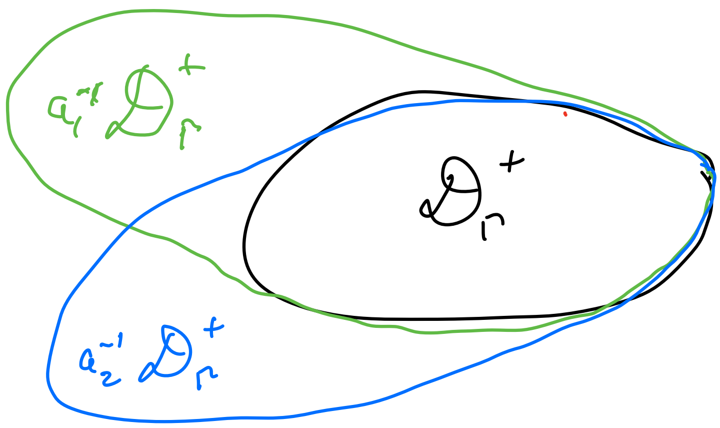
II. Cayley graphs

Joint with Boyd, Morris-Wright, & Rees

Say $a \in A_\Gamma^+$, then

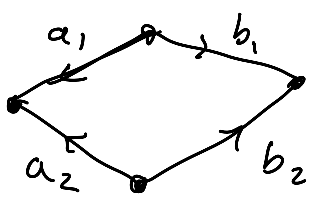
$$a\mathcal{D}_\Gamma^+ \subset \mathcal{D}_\Gamma^+ \quad \text{and} \quad \mathcal{D}_\Gamma^+ \subset a^{-1}\mathcal{D}_\Gamma^+$$

Now say $a_1, a_2 \in A_\Gamma^+$



What is $a_1^{-1}\mathcal{D}_\Gamma^+ \cap a_2^{-1}\mathcal{D}_\Gamma^+$. Is it just $\mathcal{D}_\Gamma^+$? Is it contractible? Say

$$a_1^{-1}b_1 = a_2^{-1}b_2 \quad \text{for some } b_1, b_2 \in A_\Gamma^+$$



When can this happen?

Can translate questions of this type into questions about the Cayley graph

$$\mathrm{Cay}_+(A_\Gamma) := \mathrm{Cay}(A_\Gamma, A_\Gamma^+)$$

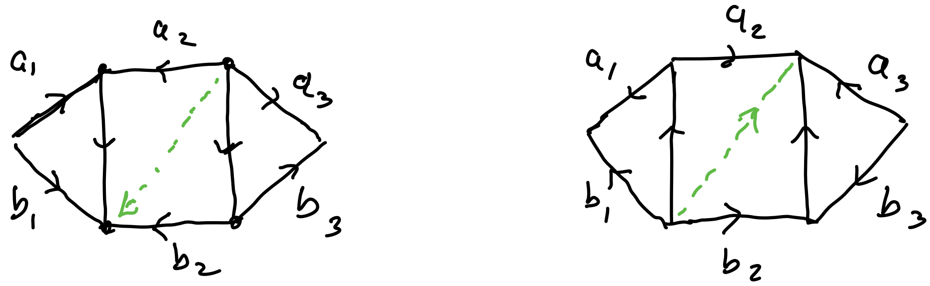
Questions / conjectures

- Conj: $Cay_+(A_\Gamma)$ has finite diameter $\iff A_\Gamma$ is spherical type
- When is $Cay_+(A_\Gamma)$ hyperbolic?
- If $T \subseteq S$, is the inclusion
$$Cay_+(A_T) \hookrightarrow Cay_+(A_\Gamma)$$
an isometric embedding?
- What conditions on $Cay_+(A_\Gamma)$ would we need to show $\mathcal{D}_\Gamma$ is contractible?

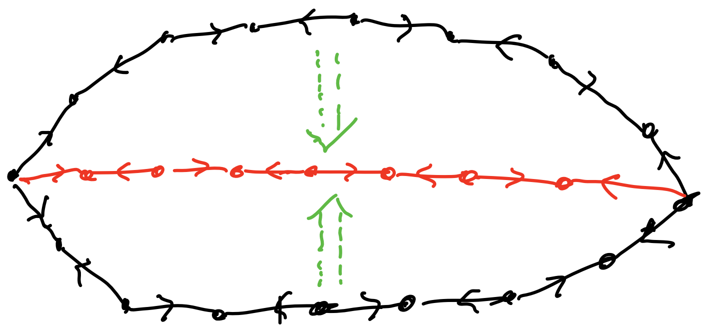
Understanding the structure of $Cay_+(A_\Gamma)$ is related to Dehornoy's work on multifraction reduction. (Joint with F. Wehrung, D. Holt, S. Rees)

multifraction $=$ alternating word in A_Γ^+
$$a_1 a_2^{-1} a_3 a_4^{-1} \cdots\cdots$$
$=$ paths in $Cay_+(A_\Gamma)$

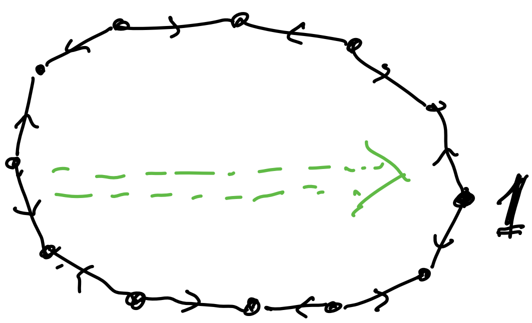
Dehornoy defines a rewriting system for such multifractions.



This rewriting system is convergent each multi-fraction rewrites to a unique irreducible sequence.



This rewriting system is semi-convergent if each multi-fraction representing $1 \in A_P$ rewrites to 1.



Thm (Dehornoy) A_Γ is convergent $\Longleftrightarrow$ A_Γ is spherical or FC-type.

Conj (Dehornoy): Every Artin group is semi-convergent.

Question: Can semi-convergence be used to answer any of the questions above?

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The End