Braiding trees: a new family of Thompson-like groups

María Cumplido Cabello
(Joint work with Julio Aroca)
Conference in the memory of Patrick Dehornoy

9th September 2021

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 205 (2006) 354–409

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

The group of parenthesized braids

Patrick Dehornoy

Laboratoire de Mathématiques Nicolas Oresme UMR 6139, Université de Caen, 14032 Caen, France

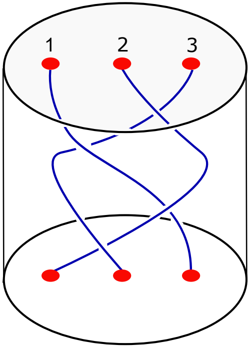
Received 31 January 2005; accepted 19 July 2005

Communicated by Alain Lascoux
Available online 21 September 2005

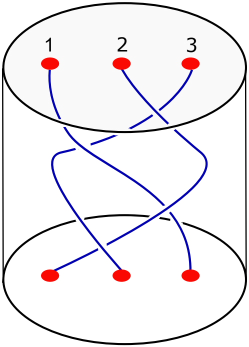
Abstract

We investigate a group $B_\bullet$ that includes Artin's braid group B_∞ and Thompson's group F. The elements of $B_\bullet$ are represented by braids diagrams in which the distances between the strands are not uniform and, besides the usual crossing generators, new rescaling operators shrink or stretch the distances between the strands. We prove that $B_\bullet$ is a group of fractions, that it is orderable, admits a nontrivial self-distributive structure, i.e., one involving the law $x(yz) = (xy)(xz)$, it embeds in the mapping class group of a sphere with a Cantor set of punctures, and that Artin's representation of B_∞ into the automorphisms of a free group extends to $B_\bullet$.

Braids

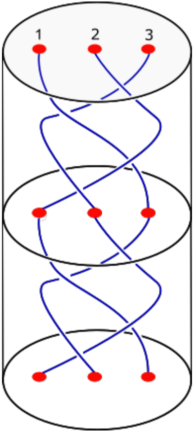


Braids

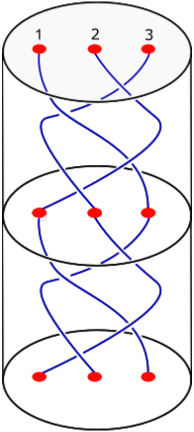


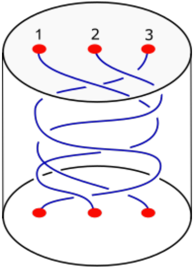
Two braids are equivalent if we can continuously deform one into the other by fixing their end points, with the condition that strands cannot touch each other.

Product of two braids

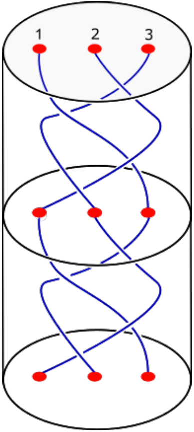


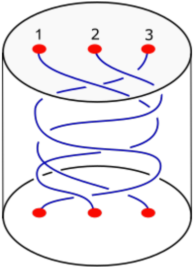
Product of two braids





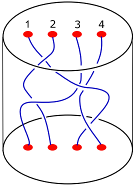
Product of two braids



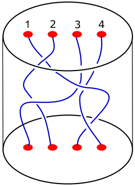


The set of equivalence classes of braids with n strands together with this product is a group, B_n.

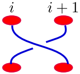
Presentation of the braid group $\mathcal{B}_n$



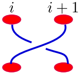
Presentation of the braid group $\mathcal{B}_n$



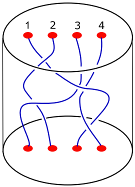
σ_i



σ_i^{-1}

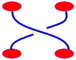


Presentation of the braid group $\mathcal{B}_n$



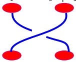
σ_i

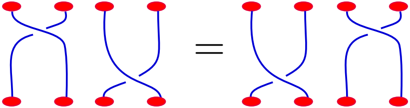
i $\quad$ $i+1$



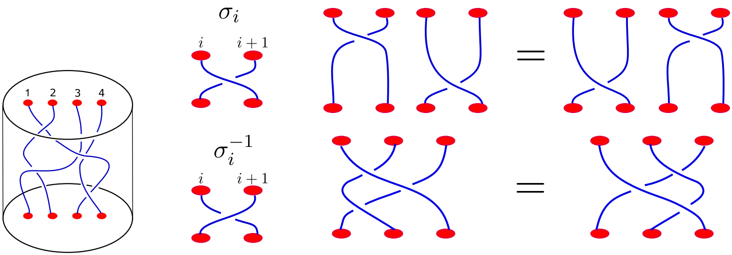
σ_i^{-1}

i $\quad$ $i+1$

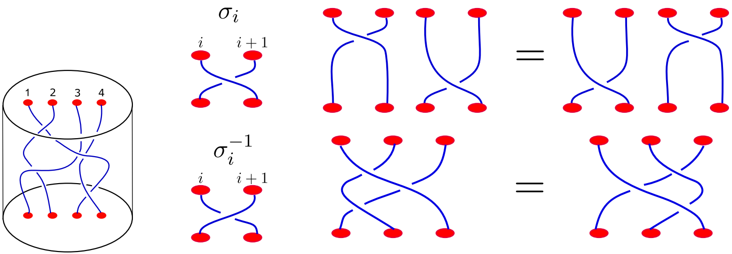




Presentation of the braid group $\mathcal{B}_n$

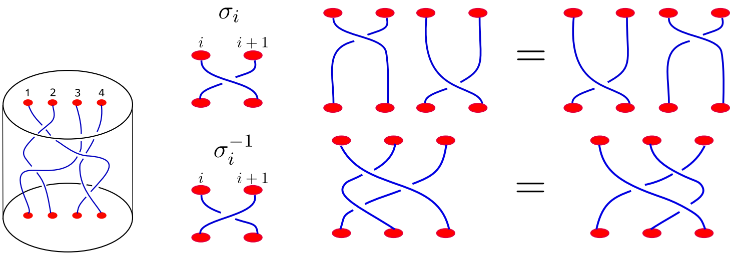


Presentation of the braid group $\mathcal{B}_n$



$$\mathcal{B}_n = \left\langle \sigma_1,\dots,\sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j=\sigma_j\sigma_i, & |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & i=1,\dots,n-2 \end{array} \right\rangle$$

Presentation of the braid group $\mathcal{B}_n$



$$\mathcal{B}_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & i = 1, \ldots, n-2 \end{array} \right\rangle$$

Coverings of the cantor set and trees

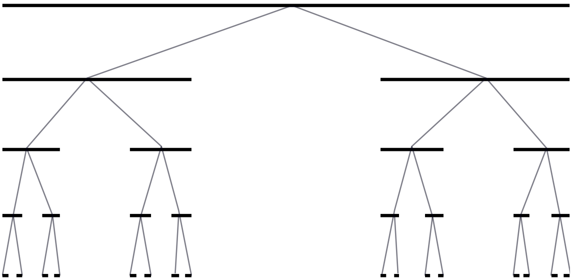
n-adic Cantor set $\mathfrak{C}_n$

Coverings of the cantor set and trees

n-adic Cantor set $\mathfrak{C}_n$

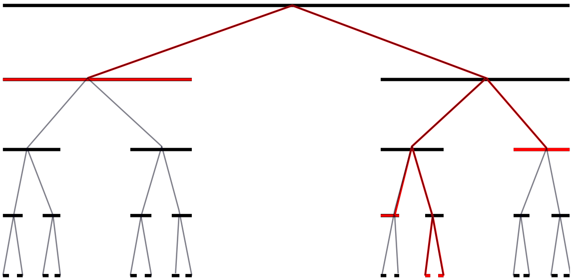
Coverings of the cantor set and trees

n-adic Cantor set $\mathfrak{C}_n$



Coverings of the cantor set and trees

n-adic Cantor set $\mathfrak{C}_n$



Covers of $\mathfrak{C}_n \leftrightarrow$ rooted subtrees of the infinite n-regular tree

Higman-Thompson's groups $V_{n,r}$

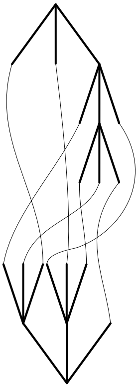
The elements of the group V_n are homeomorphisms between pairs of covers of $\mathfrak{C}_n$,

Higman-Thompson's groups $V_{n,r}$

The elements of the group V_n are homeomorphisms between pairs of covers of $\mathfrak{C}_n$, that is, bijections between the leaves of any two full n-ary trees with the same number of leaves:

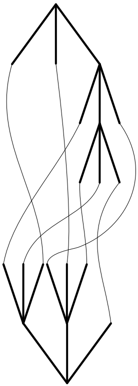
Higman-Thompson's groups $V_{n,r}$

The elements of the group V_n are homeomorphisms between pairs of covers of $\mathfrak{C}_n$, that is, bijections between the leaves of any two full n-ary trees with the same number of leaves:

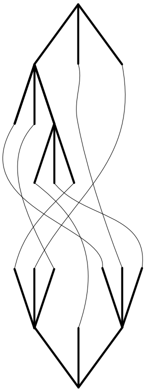


Higman-Thompson's groups $V_{n,r}$

The elements of the group V_n are homeomorphisms between pairs of covers of $\mathfrak{C}_n$, that is, bijections between the leaves of any two full n-ary trees with the same number of leaves:

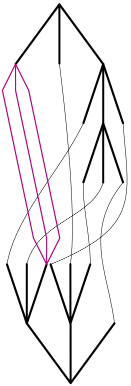


$\circ$

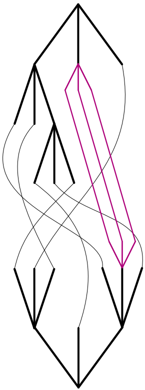


Higman-Thompson's groups $V_{n,r}$

The elements of the group V_n are homeomorphisms between pairs of covers of $\mathfrak{C}_n$, that is, bijections between the leaves of any two full n-ary trees with the same number of leaves:

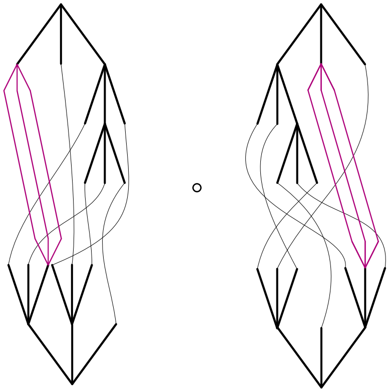


$\circ$



Higman-Thompson's groups $V_{n,r}$

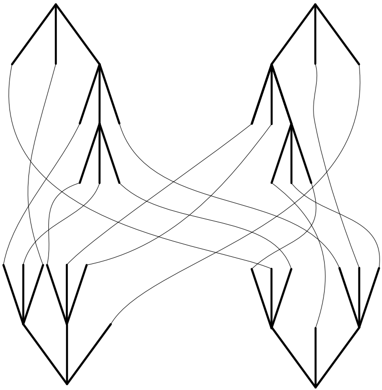
The elements of the group V_n are homeomorphisms between pairs of covers of $\mathfrak{C}_n$, that is, bijections between the leaves of any two full n-ary trees with the same number of leaves:



If we consider r copies of $\mathfrak{C}_n$ and apply the same definition, we obtain the group $V_{n,r}$, which elements are bijections between the leaves of any two r-forests of full n-ary trees with the same number of leaves.

Higman-Thompson's groups $V_{n,r}$

The elements of the group V_n are homeomorphisms between pairs of covers of $\mathfrak{C}_n$, that is, bijections between the leaves of any two full n-ary trees with the same number of leaves:



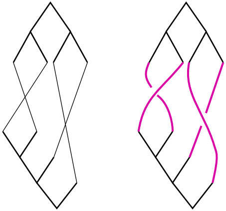
If we consider r copies of $\mathfrak{C}_n$ and apply the same definition, we obtain the group $V_{n,r}$, which elements are bijections between the leaves of any two r-forests of full n-ary trees with the same number of leaves.

Braided Thompson's group BV_2

The group BV_2 is the one that we obtain when the bijections in V_2 between the leaves of two full binary trees are replace by braids:

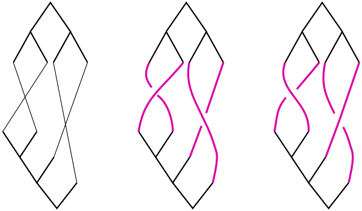
Braided Thompson's group BV_2

The group BV_2 is the one that we obtain when the bijections in V_2 between the leaves of two full binary trees are replace by braids:



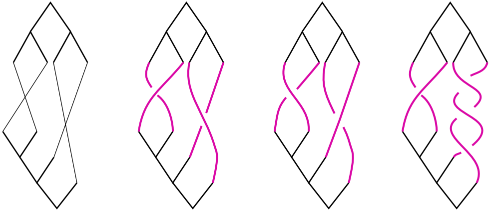
Braided Thompson's group BV_2

The group BV_2 is the one that we obtain when the bijections in V_2 between the leaves of two full binary trees are replace by braids:



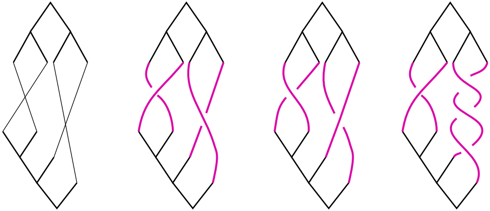
Braided Thompson's group BV_2

The group BV_2 is the one that we obtain when the bijections in V_2 between the leaves of two full binary trees are replace by braids:



Braided Thompson's group BV_2

The group BV_2 is the one that we obtain when the bijections in V_2 between the leaves of two full binary trees are replace by braids:



• This group was independently introduced by Matthew Brin and Patrick Dehornoy in 2006. They both showed that BV_2 is finitely presented and gave an explicit presentation.

What about $BV_{n,r}$?

- Indeed, the group $BV_{n,r}$ can be defined in the same way as BV_2.

What about $BV_{n,r}$?

- Indeed, the group $BV_{n,r}$ can be defined in the same way as BV_2.

- However, the definitions and techniques used by Brin and Dehornoy are rather algebraic.

What about $BV_{n,r}$?

• Indeed, the group $BV_{n,r}$ can be defined in the same way as BV_2.

• However, the definitions and techniques used by Brin and Dehornoy are rather algebraic.

- Even though it seems plausible to apply this techniques to $BV_{n,r}$, it would be extremely tedious.

What about $BV_{n,r}$?

• Indeed, the group $BV_{n,r}$ can be defined in the same way as BV_2.

• However, the definitions and techniques used by Brin and Dehornoy are rather algebraic.

- Even though it seems plausible to apply this techniques to $BV_{n,r}$, it would be extremely tedious.
- Even proving that $BV_{n,r}$ is a group would be heavy!

What about $BV_{n,r}$?

- Indeed, the group $BV_{n,r}$ can be defined in the same way as BV_2.

- However, the definitions and techniques used by Brin and Dehornoy are rather algebraic.
  - Even though it seems plausible to apply this techniques to $BV_{n,r}$, it would be extremely tedious.
  - Even proving that $BV_{n,r}$ is a group would be heavy!

- Luckily, in the last decade, new combinatorial and topological methods have been introduced.

What about $BV_{n,r}$?

- Indeed, the group $BV_{n,r}$ can be defined in the same way as BV_2.

- However, the definitions and techniques used by Brin and Dehornoy are rather algebraic.
  - Even though it seems plausible to apply this techniques to $BV_{n,r}$, it would be extremely tedious.
  - Even proving that $BV_{n,r}$ is a group would be heavy!

- Luckily, in the last decade, new combinatorial and topological methods have been introduced.
  - In our paper, we generalise BV_2 to a much larger family of groups $BV_{n,r}(H), H \leq \mathcal{B}_n$ and we use new approaches to prove that they are groups and give a finite set of generators if H is finitely generated.

Recursive braids

A recursive α-braid, for α in some $\mathcal{B}_m$, is a braid of infinite strands constructed from one strand as follows:

1. Split the strand in m strands and braid them as α indicates.
2. Repeat this process in every strand on the new braid.
3. Repeat to the infinity.

Recursive braids

A recursive α-braid, for α in some $\mathcal{B}_m$, is a braid of infinite strands constructed from one strand as follows:

1. Split the strand in m strands and braid them as α indicates.
2. Repeat this process in every strand on the new braid.
3. Repeat to the infinity.

Example for $\alpha = \sigma_1 \in \mathcal{B}_2$:

Recursive braids

A recursive α-braid, for α in some $\mathcal{B}_m$, is a braid of infinite strands constructed from one strand as follows:

1. Split the strand in m strands and braid them as α indicates.
2. Repeat this process in every strand on the new braid.
3. Repeat to the infinity.

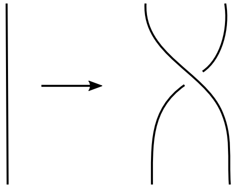
Example for $\alpha = \sigma_1 \in \mathcal{B}_2$:

Recursive braids

A recursive α-braid, for α in some $\mathcal{B}_m$, is a braid of infinite strands constructed from one strand as follows:

1. Split the strand in m strands and braid them as α indicates.
2. Repeat this process in every strand on the new braid.
3. Repeat to the infinity.

Example for $\alpha = \sigma_1 \in \mathcal{B}_2$:

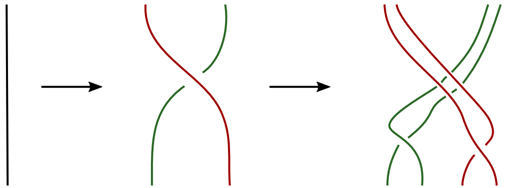


Recursive braids

A recursive α-braid, for α in some $\mathcal{B}_m$, is a braid of infinite strands constructed from one strand as follows:

1. Split the strand in m strands and braid them as α indicates.
2. Repeat this process in every strand on the new braid.
3. Repeat to the infinity.

Example for $\alpha = \sigma_1 \in \mathcal{B}_2$:

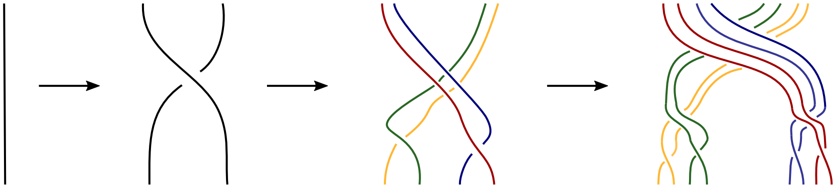


Recursive braids

A recursive α-braid, for α in some $\mathcal{B}_m$, is a braid of infinite strands constructed from one strand as follows:

1. Split the strand in m strands and braid them as α indicates.
2. Repeat this process in every strand on the new braid.
3. Repeat to the infinity.

Example for $\alpha = \sigma_1 \in \mathcal{B}_2$:



Definition $BV_{n,r}(H)$

Definition [Aroca & C. 2020]

Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

Definition $BV_{n,r}(H)$

Definition [Aroca & C. 2020]

Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

• In the diagram with trees joined by a braid, the recursive braids are represented by white vertices labelled with braids in H.

Definition $BV_{n,r}(H)$

Definition [Aroca & C. 2020]

Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

• In the diagram with trees joined by a braid, the recursive braids are represented by white vertices labelled with braids in H.

Why does this make sense?

Definition $BV_{n,r}(H)$

> Definition [Aroca & C. 2020]
>
> Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

• In the diagram with trees joined by a braid, the recursive braids are represented by white vertices labelled with braids in H.

Why does this make sense?

Because diagrams are considered up to equivalence:

Definition $BV_{n,r}(H)$

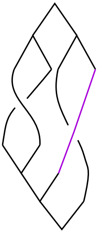
Definition [Aroca & C. 2020]

Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

• In the diagram with trees joined by a braid, the recursive braids are represented by white vertices labelled with braids in H.

Why does this make sense?

Because diagrams are considered up to equivalence:



Definition $BV_{n,r}(H)$

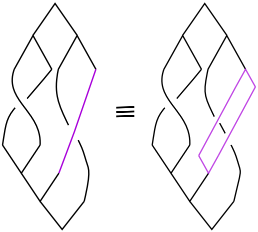
Definition [Aroca & C. 2020]

Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

• In the diagram with trees joined by a braid, the recursive braids are represented by white vertices labelled with braids in H.

Why does this make sense?

Because diagrams are considered up to equivalence:



Definition $BV_{n,r}(H)$

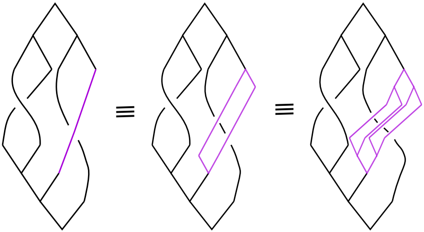
Definition [Aroca & C. 2020]

Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

• In the diagram with trees joined by a braid, the recursive braids are represented by white vertices labelled with braids in H.

Why does this make sense?

Because diagrams are considered up to equivalence:



Definition $BV_{n,r}(H)$

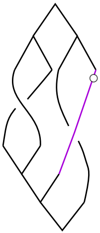
Definition [Aroca & C. 2020]

Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

• In the diagram with trees joined by a braid, the recursive braids are represented by white vertices labelled with braids in H.

Why does this make sense?

Because diagrams are considered up to equivalence:



Definition $BV_{n,r}(H)$

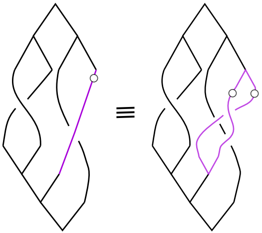
Definition [Aroca & C. 2020]

Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

• In the diagram with trees joined by a braid, the recursive braids are represented by white vertices labelled with braids in H.

Why does this make sense?

Because diagrams are considered up to equivalence:



Definition $BV_{n,r}(H)$

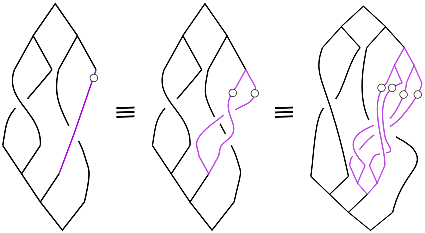
Definition [Aroca & C. 2020]

Given H a subgroup of the braid group on n strands, we define $\boldsymbol{BV_{n,r}(H)}$ as the group $BV_{n,r}$ with recursive α braids, $\alpha \in H$, between covers of $\mathfrak{C}_n$.

• In the diagram with trees joined by a braid, the recursive braids are represented by white vertices labelled with braids in H.

Why does this make sense?

Because diagrams are considered up to equivalence:



Proving that $BV_{n,r}(H)$ is a group

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is a group.

Proving that $BV_{n,r}(H)$ is a group

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is a group.

Idea of the proof.

Inspired by [Newman '42], [Belk & Matucci '14] and [Aroca '18]

Proving that $BV_{n,r}(H)$ is a group

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is a group.

Idea of the proof.

Inspired by [Newman '42], [Belk & Matucci '14] and [Aroca '18]

1. We define braided diagrams ($\supset$ (tree, braid, tree)-diagrams).

Proving that $BV_{n,r}(H)$ is a group

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is a group.

Idea of the proof.

Inspired by [Newman '42], [Belk & Matucci '14] and [Aroca '18]

1. We define braided diagrams ($\supset$ (tree, braid, tree)-diagrams).
2. We describe a series of moves that states when two braided diagrams are equivalent.

Proving that $BV_{n,r}(H)$ is a group

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is a group.

Idea of the proof.

Inspired by [Newman '42], [Belk & Matucci '14] and [Aroca '18]

1. We define braided diagrams ($\supset$ (tree, braid, tree)-diagrams).
2. We describe a series of moves that states when two braided diagrams are equivalent.
3. We show that a braided diagram is equivalent to a unique "minimal" braided diagram.

Proving that $BV_{n,r}(H)$ is a group

Theorem [Aroca & C. 2020]
$BV_{n,r}(H)$ is a group.

Idea of the proof.
Inspired by [Newman '42], [Belk & Matucci '14] and [Aroca '18]

1. We define braided diagrams ($\supset$ (tree, braid, tree)-diagrams).
2. We describe a series of moves that states when two braided diagrams are equivalent.
3. We show that a braided diagram is equivalent to a unique "minimal" braided diagram.
4. There is a bijection between classes of equivalent braided diagrams and the elements of $BV_{n,r}(H)$.

Proving that $BV_{n,r}(H)$ is a group

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is a group.

Idea of the proof.

Inspired by [Newman '42], [Belk & Matucci '14] and [Aroca '18]

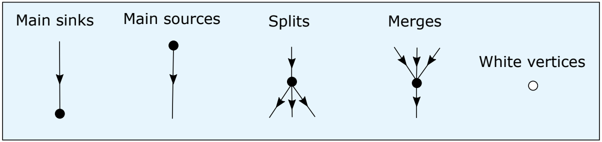
1. We define braided diagrams ($\supset$ (tree, braid, tree)-diagrams).
2. We describe a series of moves that states when two braided diagrams are equivalent.
3. We show that a braided diagram is equivalent to a unique "minimal" braided diagram.
4. There is a bijection between classes of equivalent braided diagrams and the elements of $BV_{n,r}(H)$.
5. The composition of diagrams provides a group structure.

Proving that $BV_{n,r}(H)$ is a group (First step)

A braided diagram is a (good) planar projection of a directed graph

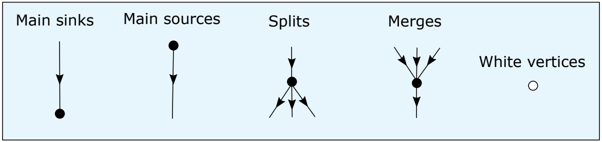
Proving that $BV_{n,r}(H)$ is a group (First step)

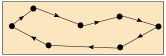
A braided diagram is a (good) planar projection of a directed graph that only contains the following types of vertices:



Proving that $BV_{n,r}(H)$ is a group (First step)

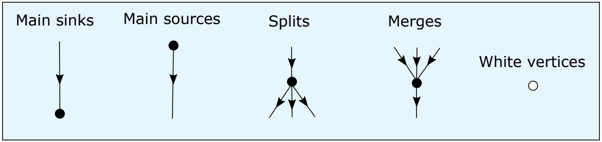
A braided diagram is a (good) planar projection of a directed graph that only contains the following types of vertices:

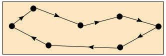


and does not contain oriented loops: 

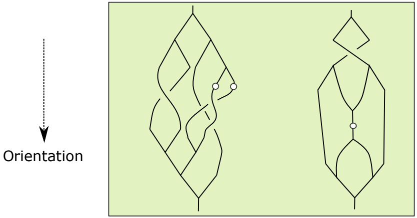
Proving that $BV_{n,r}(H)$ is a group (First step)

A braided diagram is a (good) planar projection of a directed graph that only contains the following types of vertices:



and does not contain oriented loops: 

Examples:

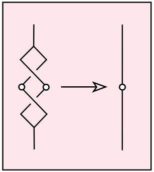


Proving that $BV_{n,r}(H)$ is a group (Second step)

Two braided diagrams are equivalent if we can transform one into the other by doing a series of the following 6 moves:

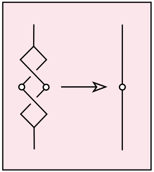
Proving that $BV_{n,r}(H)$ is a group (Second step)

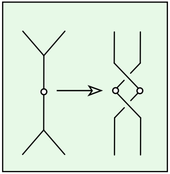
Two braided diagrams are equivalent if we can transform one into the other by doing a series of the following 6 moves:



Proving that $BV_{n,r}(H)$ is a group (Second step)

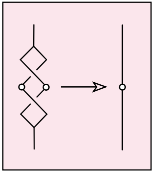
Two braided diagrams are equivalent if we can transform one into the other by doing a series of the following 6 moves:

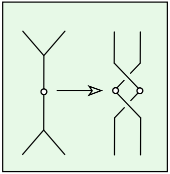


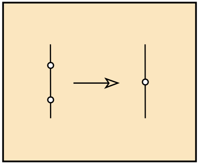


Proving that $BV_{n,r}(H)$ is a group (Second step)

Two braided diagrams are equivalent if we can transform one into the other by doing a series of the following 6 moves:

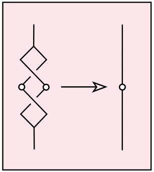


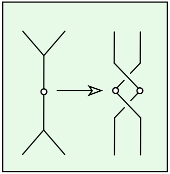


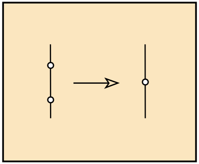


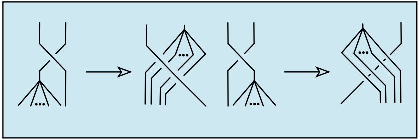
Proving that $BV_{n,r}(H)$ is a group (Second step)

Two braided diagrams are equivalent if we can transform one into the other by doing a series of the following 6 moves:



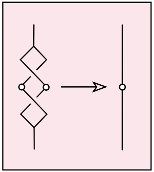


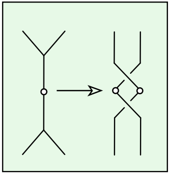


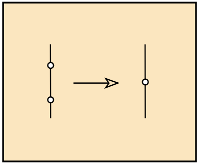


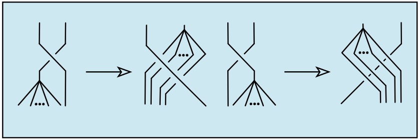
Proving that $BV_{n,r}(H)$ is a group (Second step)

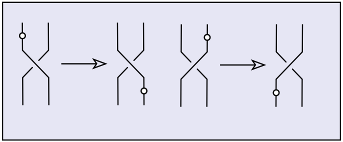
Two braided diagrams are equivalent if we can transform one into the other by doing a series of the following 6 moves:





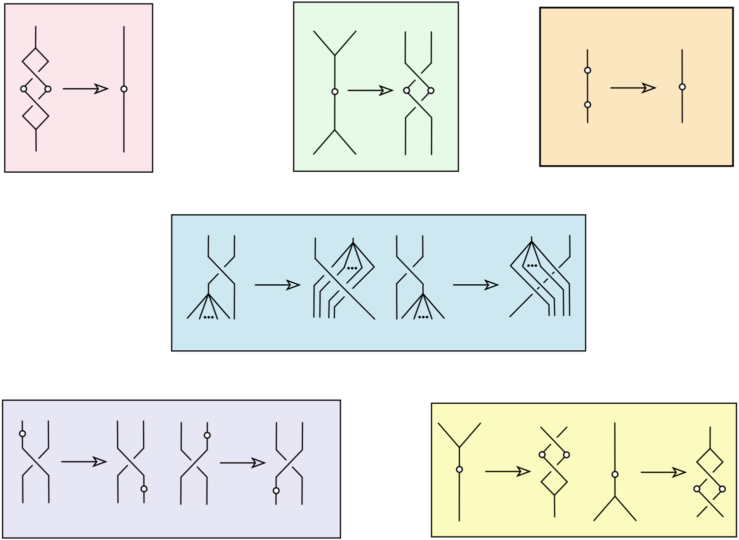






Proving that $BV_{n,r}(H)$ is a group (Second step)

Two braided diagrams are equivalent if we can transform one into the other by doing a series of the following 6 moves:



Proving that $BV_{n,r}(H)$ is a group (Third step)

A braided diagram is reduced if no moves can be performed on it.

Proving that $BV_{n,r}(H)$ is a group (Third step)

A braided diagram is reduced if no moves can be performed on it.

Lemma [Aroca & C. 2020]

Every braided diagram is equivalent to a unique reduced braided diagram.

Proving that $BV_{n,r}(H)$ is a group (Third step)

A braided diagram is reduced if no moves can be performed on it.

Lemma [Aroca & C. 2020]

Every braided diagram is equivalent to a unique reduced braided diagram.

Idea of the proof:

Proving that $BV_{n,r}(H)$ is a group (Third step)

A braided diagram is reduced if no moves can be performed on it.

Lemma [Aroca & C. 2020]

Every braided diagram is equivalent to a unique reduced braided diagram.

Idea of the proof:

- We construct a directed graph (rewriting system) as follows:

Proving that $BV_{n,r}(H)$ is a group (Third step)

A braided diagram is reduced if no moves can be performed on it.

Lemma [Aroca & C. 2020]

Every braided diagram is equivalent to a unique reduced braided diagram.

Idea of the proof:

- We construct a directed graph (rewriting system) as follows:
  - The set of vertices is the set of braided diagrams.

Proving that $BV_{n,r}(H)$ is a group (Third step)

A braided diagram is reduced if no moves can be performed on it.

Lemma [Aroca & C. 2020]

Every braided diagram is equivalent to a unique reduced braided diagram.

Idea of the proof:

- We construct a directed graph (rewriting system) as follows:
  - The set of vertices is the set of braided diagrams.
  - There is an edge from a braided diagram D to another D' if D' can be obtained from the by performing a move in D.

Proving that $BV_{n,r}(H)$ is a group (Third step)

A braided diagram is reduced if no moves can be performed on it.

Lemma [Aroca & C. 2020]

Every braided diagram is equivalent to a unique reduced braided diagram.

Idea of the proof:

- We construct a directed graph (rewriting system) as follows:
  - The set of vertices is the set of braided diagrams.
  - There is an edge from a braided diagram D to another D' if D' can be obtained from the by performing a move in D.
- We prove that the rewriting system satisfies the following properties:

Proving that $BV_{n,r}(H)$ is a group (Third step)

A braided diagram is reduced if no moves can be performed on it.

Lemma [Aroca & C. 2020]

Every braided diagram is equivalent to a unique reduced braided diagram.

Idea of the proof:

- We construct a directed graph (rewriting system) as follows:
  - The set of vertices is the set of braided diagrams.
  - There is an edge from a braided diagram D to another D' if D' can be obtained from the by performing a move in D.
- We prove that the rewriting system satisfies the following properties:
  - It is terminating: every oriented path is finite.

Proving that $BV_{n,r}(H)$ is a group (Third step)

A braided diagram is reduced if no moves can be performed on it.

Lemma [Aroca & C. 2020]

Every braided diagram is equivalent to a unique reduced braided diagram.

Idea of the proof:

- We construct a directed graph (rewriting system) as follows:
  - The set of vertices is the set of braided diagrams.
  - There is an edge from a braided diagram D to another D' if D' can be obtained from the by performing a move in D.
- We prove that the rewriting system satisfies the following properties:
  - It is terminating: every oriented path is finite.
  - It is locally confluent: If D_1 and D_2 are reductions of a diagram D, then there exists D' which is a reduction of both D_1 and D_2.

Proving that $BV_{n,r}(H)$ is a group (Fourth step)

Lemma [Aroca & C. 2020]

There is a bijection between classes of equivalent braided diagrams and classes of equivalent elements of $BV_{n,r}(H)$.

Proving that $BV_{n,r}(H)$ is a group (Fourth step)

Lemma [Aroca & C. 2020]

There is a bijection between classes of equivalent braided diagrams and classes of equivalent elements of $BV_{n,r}(H)$.

Idea of the proof.

Proving that $BV_{n,r}(H)$ is a group (Fourth step)

Lemma [Aroca & C. 2020]

There is a bijection between classes of equivalent braided diagrams and classes of equivalent elements of $BV_{n,r}(H)$.

Idea of the proof.

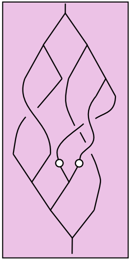
- Elements of $BV_{n,r}(H)$ are represented by reduced braided diagrams.

Proving that $BV_{n,r}(H)$ is a group (Fourth step)

Lemma [Aroca & C. 2020]

There is a bijection between classes of equivalent braided diagrams and classes of equivalent elements of $BV_{n,r}(H)$.

Idea of the proof.



- Elements of $BV_{n,r}(H)$ are represented by reduced braided diagrams.

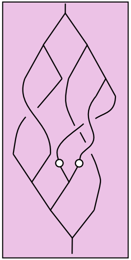
(We can not perform any move once we have pushed all white vertices to the bottom tree).

Proving that $BV_{n,r}(H)$ is a group (Fourth step)

Lemma [Aroca & C. 2020]

There is a bijection between classes of equivalent braided diagrams and classes of equivalent elements of $BV_{n,r}(H)$.

Idea of the proof.



- Elements of $BV_{n,r}(H)$ are represented by reduced braided diagrams.

(We can not perform any move once we have pushed all white vertices to the bottom tree).

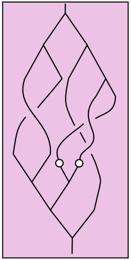
- Reduced braided diagrams are diagrams (tree, braid, tree) with all the vertices lying on the leaves of the bottom tree.

Proving that $BV_{n,r}(H)$ is a group (Fourth step)

Lemma [Aroca & C. 2020]

There is a bijection between classes of equivalent braided diagrams and classes of equivalent elements of $BV_{n,r}(H)$.

Idea of the proof.



- Elements of $BV_{n,r}(H)$ are represented by reduced braided diagrams.

(We can not perform any move once we have pushed all white vertices to the bottom tree).

- Reduced braided diagrams are diagrams (tree, braid, tree) with all the vertices lying on the leaves of the bottom tree.

(We study the oriented path in a reduced diagram from a main source to a main sink).

Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.

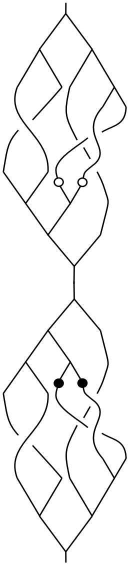
Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

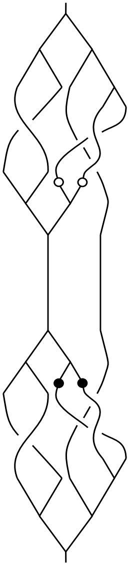
  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

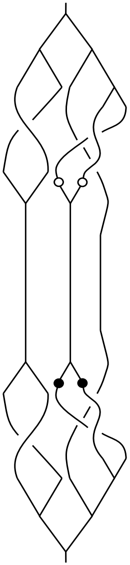
  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

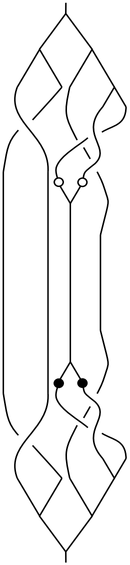
  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

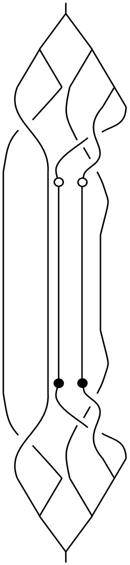
  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

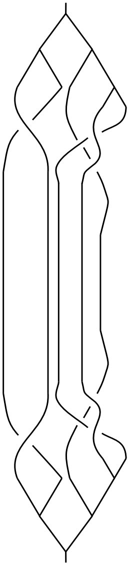
  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

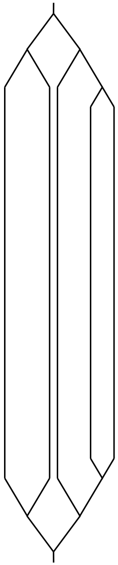
  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

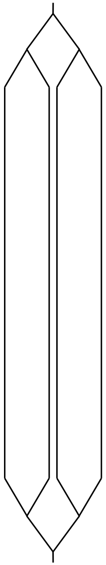
  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

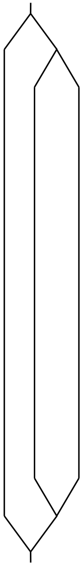
  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

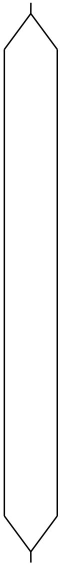
  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:



Proving that $BV_{n,r}(H)$ is a group (Last step)

- The classes of braided diagrams with the diagram composition have a group structure.
- The previous bijection is a group homomorphism.

  The composition of two elements can be seen as the composition of the corresponding reduced braided diagrams:

Finite generation

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is finitely generated if H is finitely generated.

Finite generation

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is finitely generated if H is finitely generated.

We take the strategy that Higman used in 1974 to prove that V_n is finitely generated and modify it to include braids and white vertices.

Finite generation

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is finitely generated if H is finitely generated.

We take the strategy that Higman used in 1974 to prove that V_n is finitely generated and modify it to include braids and white vertices.

The depth of a full n-ary tree is the number of  that it contains.

Finite generation

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is finitely generated if H is finitely generated.

We take the strategy that Higman used in 1974 to prove that V_n is finitely generated and modify it to include braids and white vertices.

The depth of a full n-ary tree is the number of  that it contains.

The depth of an element in $BV_{n,r}(H)$ is the depth of the trees of it reduced braided diagram.

Finite generation

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is finitely generated if H is finitely generated.

We take the strategy that Higman used in 1974 to prove that V_n is finitely generated and modify it to include braids and white vertices.

The depth of a full n-ary tree is the number of  that it contains.

The depth of an element in $BV_{n,r}(H)$ is the depth of the trees of it reduced braided diagram.

What Higman proves for V_n is that one can decompose an element of depth $d \geq 3$ using elements of depth $< d$. We do a similar thing for our groups.

Finite generation

Theorem [Aroca & C. 2020]

$BV_{n,r}(H)$ is finitely generated if H is finitely generated.

We take the strategy that Higman used in 1974 to prove that V_n is finitely generated and modify it to include braids and white vertices.

The depth of a full n-ary tree is the number of  that it contains.

The depth of an element in $BV_{n,r}(H)$ is the depth of the trees of it reduced braided diagram.

What Higman proves for V_n is that one can decompose an element of depth $d \geq 3$ using elements of depth $< d$. We do a similar thing for our groups.

Finite generation

Proposition [Aroca & C. 2020]

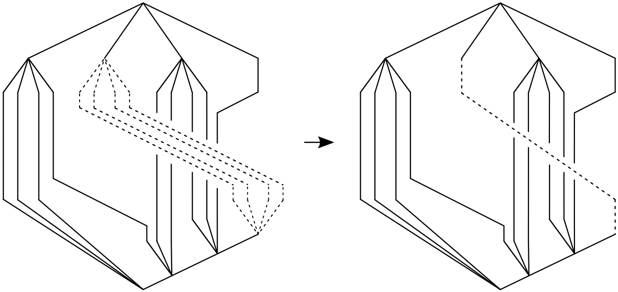
Every element of $BV_{n,r}(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_{n,r}(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. g.).

Finite generation

Proposition [Aroca & C. 2020]

Every element of $BV_{n,r}(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_{n,r}(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. g.).

Usage of ribbons



Finite generation

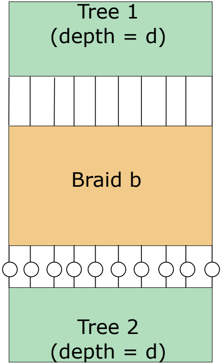
Proposition [Aroca & C. 2020]

Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).

Finite generation

Proposition [Aroca & C. 2020]

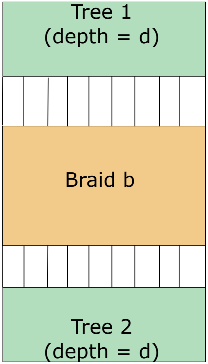
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).



Finite generation

Proposition [Aroca & C. 2020]

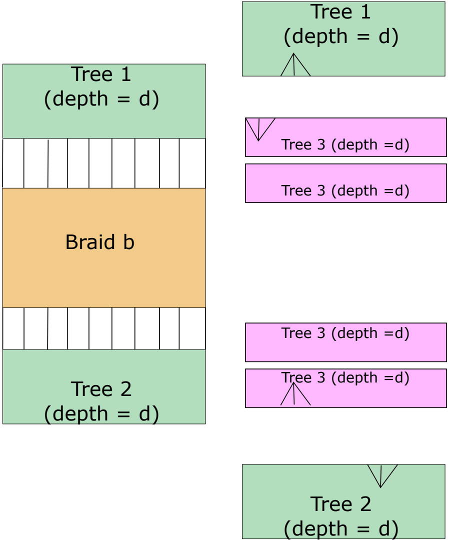
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).



Finite generation

Proposition [Aroca & C. 2020]

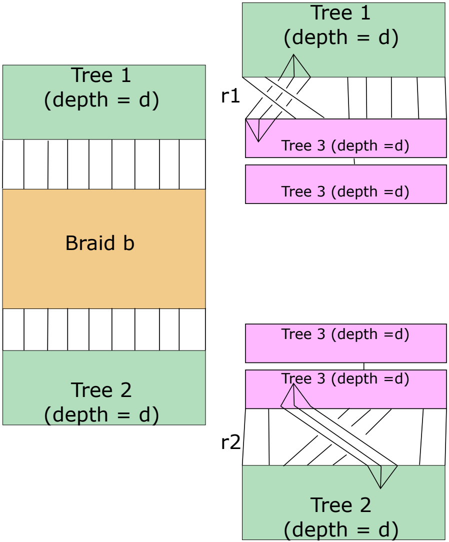
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).



Finite generation

Proposition [Aroca & C. 2020]

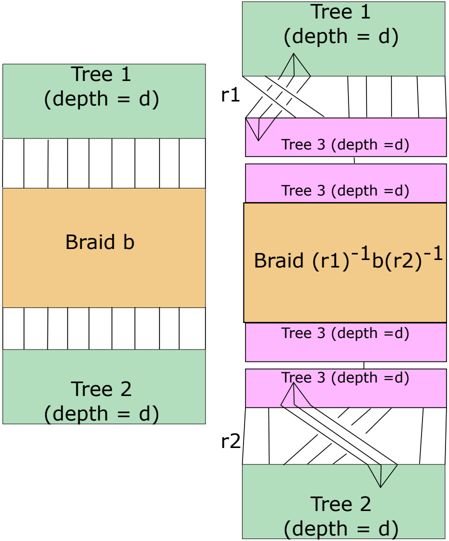
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).



Finite generation

Proposition [Aroca & C. 2020]

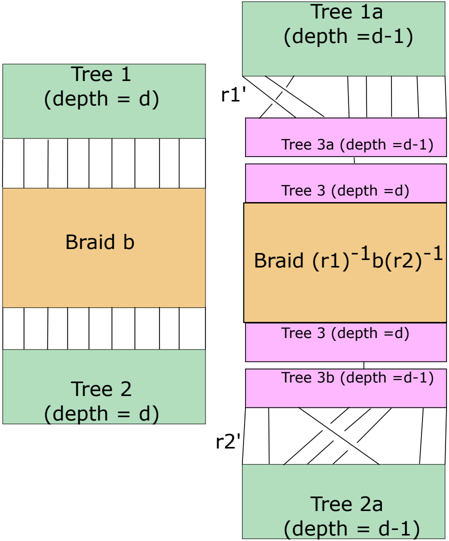
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).



Finite generation

Proposition [Aroca & C. 2020]

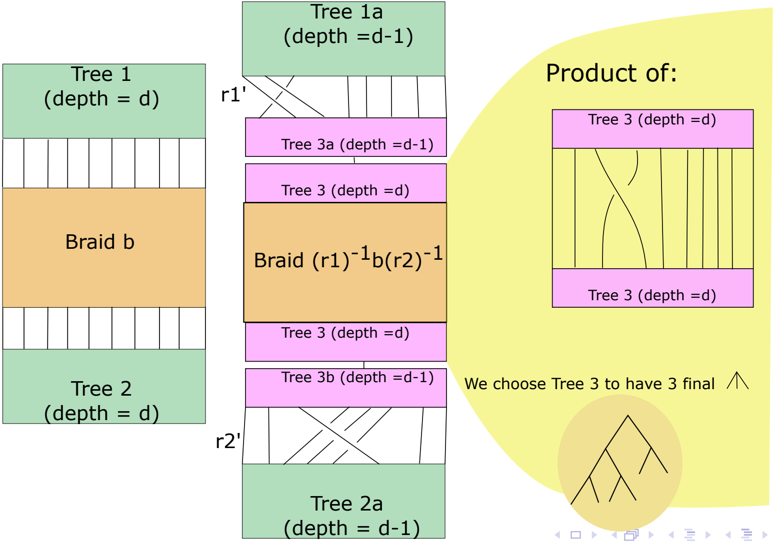
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).



Finite generation

Proposition [Aroca & C. 2020]

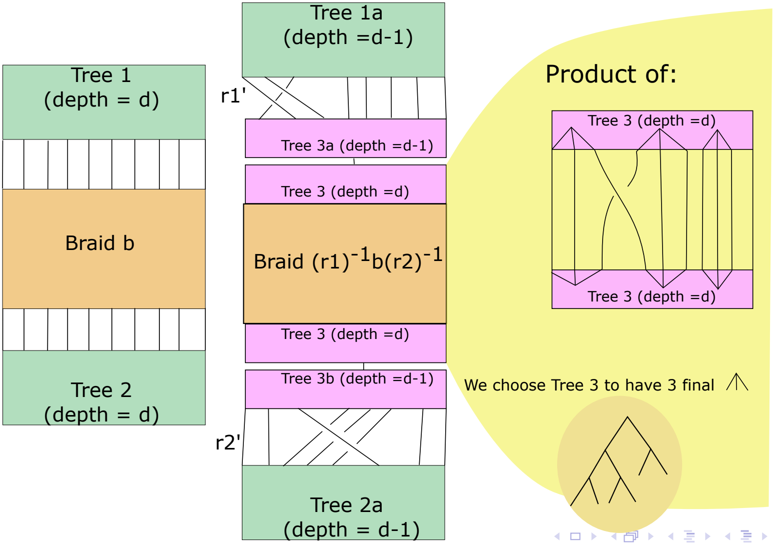
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).



Finite generation

Proposition [Aroca & C. 2020]

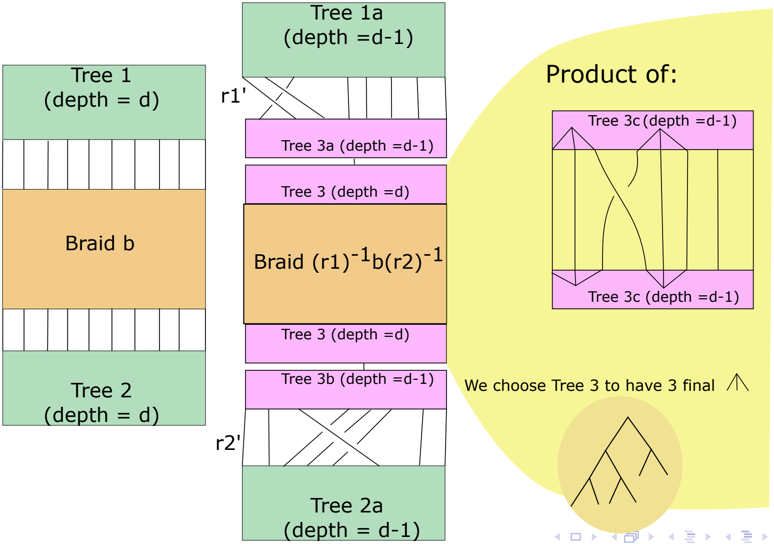
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).



Finite generation

Proposition [Aroca & C. 2020]

Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_{n,r}(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

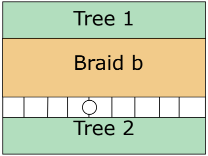
Proposition [Aroca & C. 2020]

Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).

Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

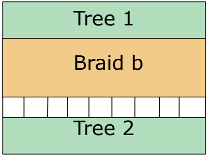
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

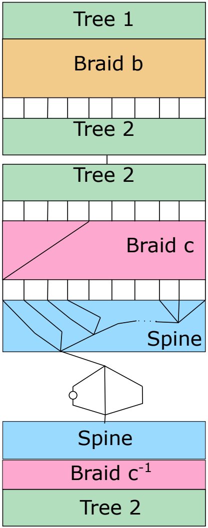
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

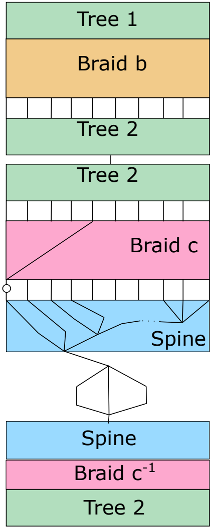
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

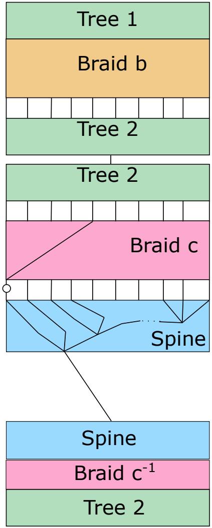
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

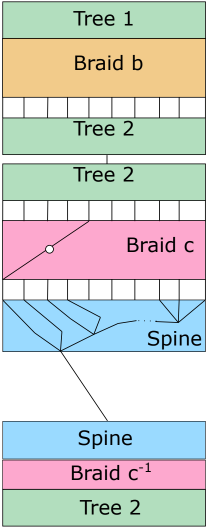
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

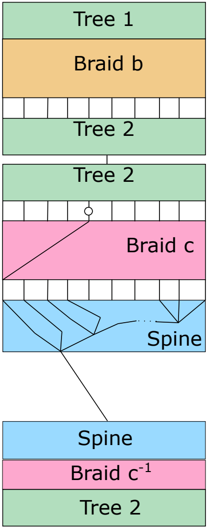
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

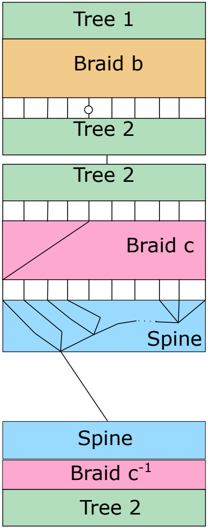
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

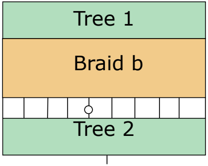
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

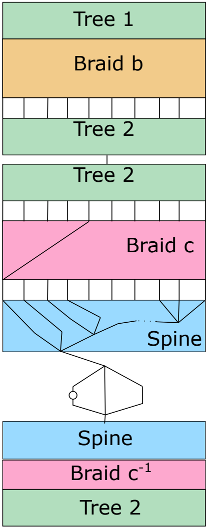
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

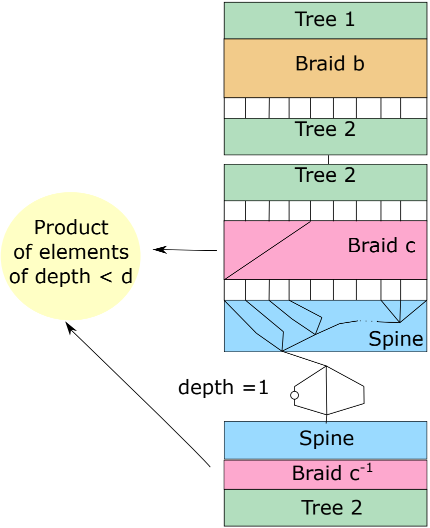
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

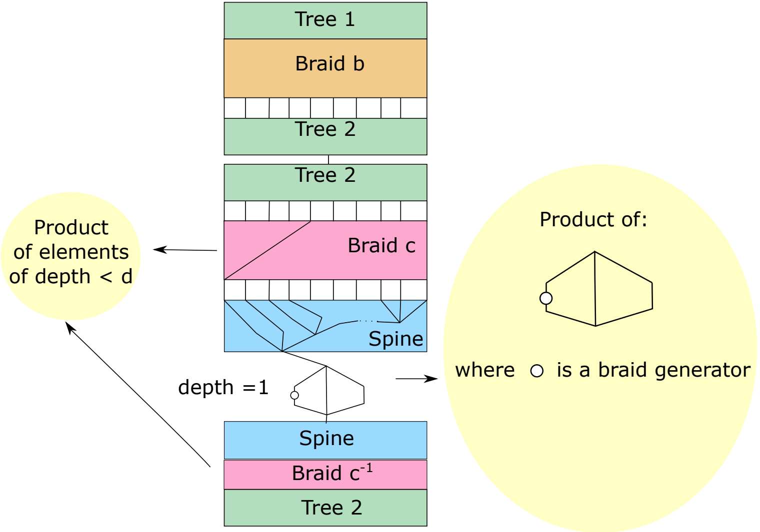
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

Proposition [Aroca & C. 2020]

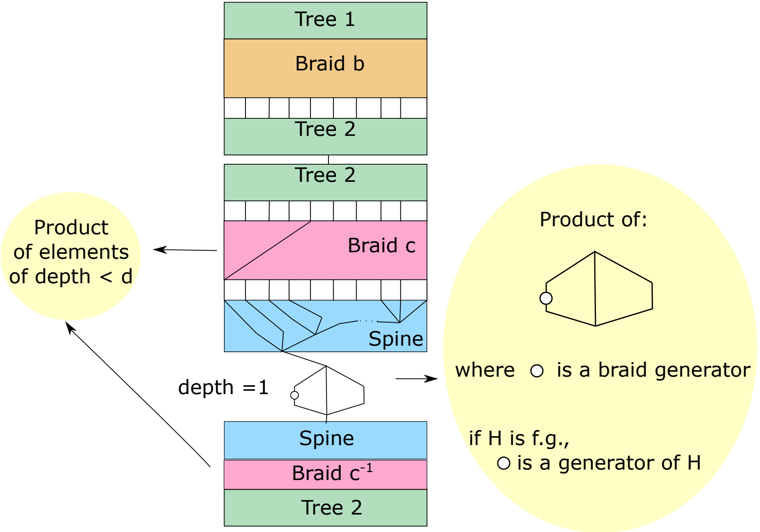
Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).



Finite generation (dealing with white vertices)

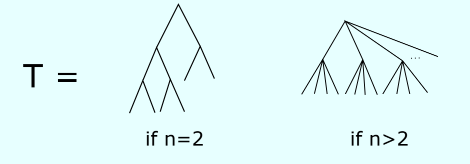
Proposition [Aroca & C. 2020]

Every element of $BV_n(H)$ of depth $d > 4$ can be expressed as the product of elements in $BV_n(\mathcal{B}_n)$ of depth $< d$ (in $BV_n(H)$ if H is f. generated).

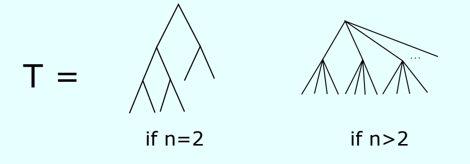


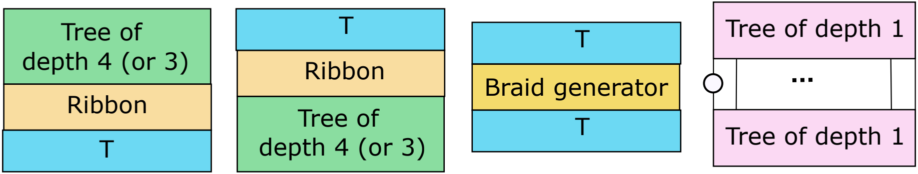
(Big) set of generators for $BV_n(H)$

(Big) set of generators for $BV_n(H)$



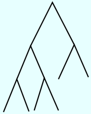
(Big) set of generators for $BV_n(H)$



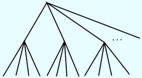


(Big) set of generators for $BV_n(H)$

T =



if n=2



if n>2

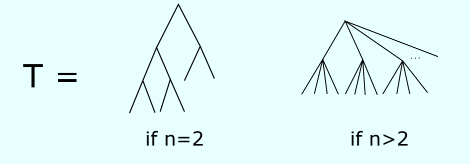
Tree of depth 4 (or 3)
Ribbon
T

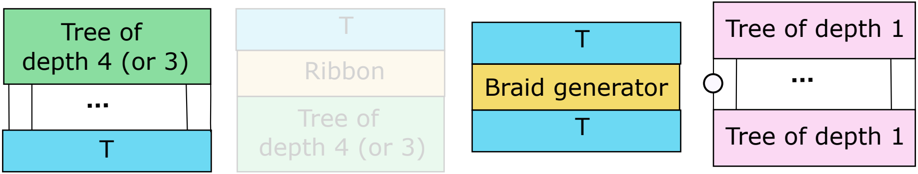
T
Ribbon
Tree of depth 4 (or 3)

T
Braid generator
T

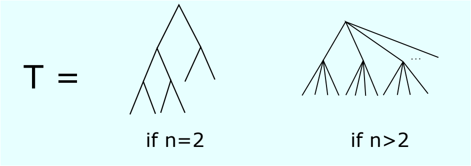
Tree of depth 1
...
Tree of depth 1

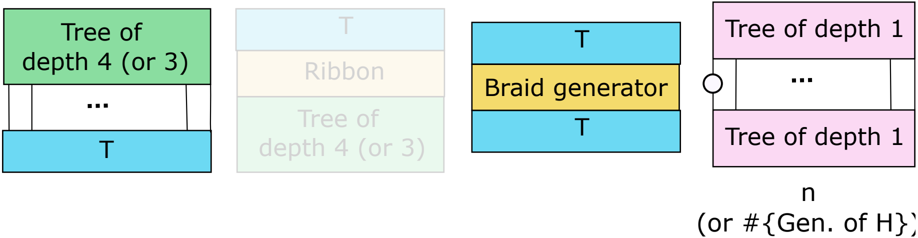
(Big) set of generators for $BV_n(H)$



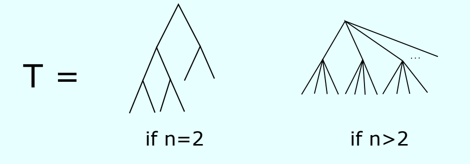


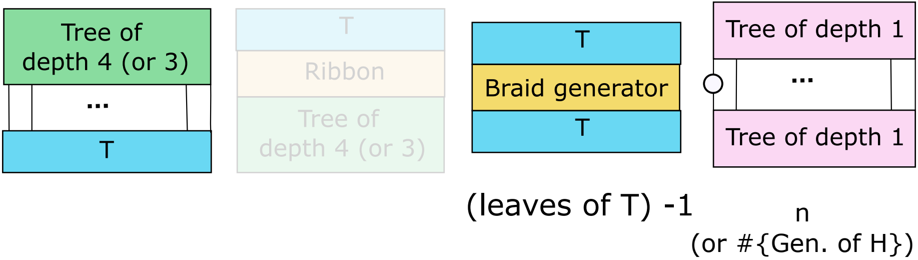
(Big) set of generators for $BV_n(H)$



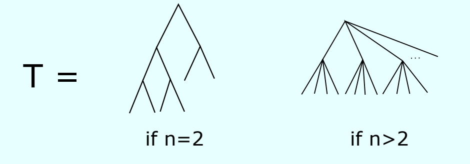


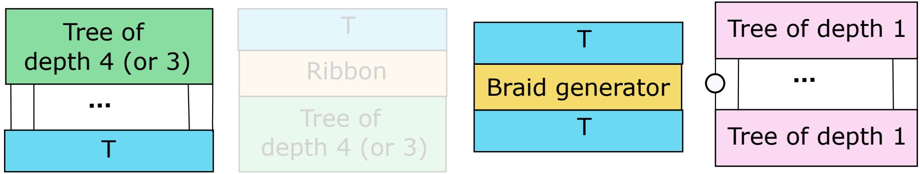
(Big) set of generators for $BV_n(H)$





(Big) set of generators for $BV_n(H)$



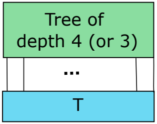


finite
(but many!)

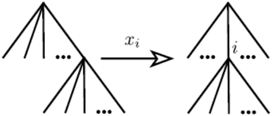
(leaves of T) -1

n
(or #{Gen. of H})

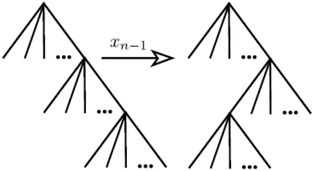
Reducing the set of generators



Thanks to [Brown, 1987] we know
that these generators can be expressed
as the product of the following n elements:



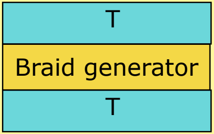
$i \in \{0, \ldots, n-2\}$



Reducing the set of generators (only when $H = \mathcal{B}_n$)

Lemma [Aroca & C. 2020]

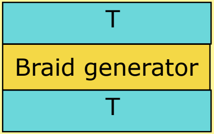
We just need one of these generators



Reducing the set of generators (only when $H = \mathcal{B}_n$)

Lemma [Aroca & C. 2020]

We just need one of these generators

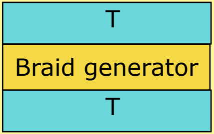


The proof is based in two ideas:

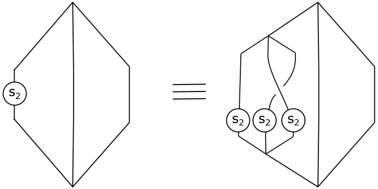
Reducing the set of generators (only when $H = \mathcal{B}_n$)

Lemma [Aroca & C. 2020]

We just need one of these generators



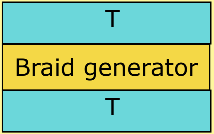
The proof is based in two ideas:



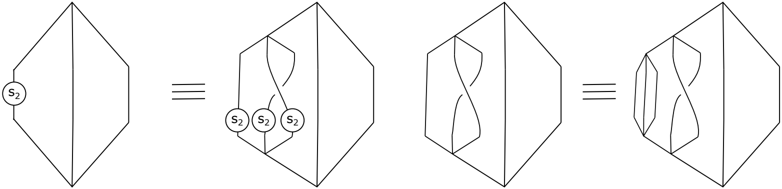
Reducing the set of generators (only when $H = \mathcal{B}_n$)

Lemma [Aroca & C. 2020]

We just need one of these generators



The proof is based in two ideas:



Set of generators (when H is finitely generated)

Theorem [Aroca & C. 2020]

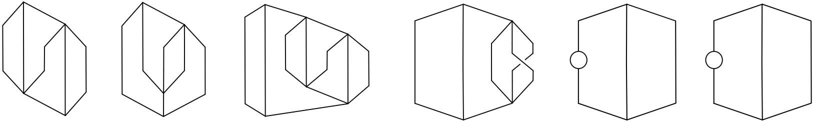
- $BV_n(\mathcal{B}_n)$ is generated by at most $2n+1$ known elements.
- $BV_n(H)$ is generated by at most $n + |\{\text{gen. of } H\}| + (\text{leaves of } T) - 1$ known elements (if the gen. of H are known).
- If H is a parabolic subgroup, we can further reduce the set of generators.

Set of generators (when H is finitely generated)

Theorem [Aroca & C. 2020]

- $BV_n(\mathcal{B}_n)$ is generated by at most $2n+1$ known elements.
- $BV_n(H)$ is generated by at most $n + |\{\text{gen. of } H\}| + (\text{leaves of } T) - 1$ known elements (if the gen. of H are known).
- If H is a parabolic subgroup, we can further reduce the set of generators.

Generators for $BV_3(\mathcal{B}_3)$:

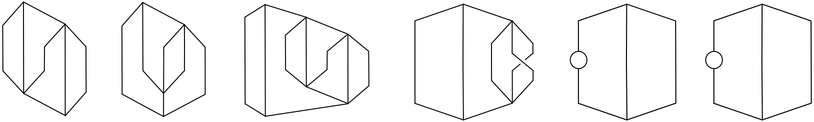


Set of generators (when H is finitely generated)

Theorem [Aroca & C. 2020]

- $BV_n(\mathcal{B}_n)$ is generated by at most $2n+1$ known elements.
- $BV_n(H)$ is generated by at most $n + |\{\text{gen. of } H\}| + (\text{leaves of } T) - 1$ known elements (if the gen. of H are known).
- If H is a parabolic subgroup, we can further reduce the set of generators.

Generators for $BV_3(\mathcal{B}_3)$:



Finally, the proof can be easily adapted for $BV_{n,r}(H)$.

Thank you!