COHERENT PRESENTATIONS OF A CLASS OF MONOIDS ADMITTING A GARSIDE FAMILY

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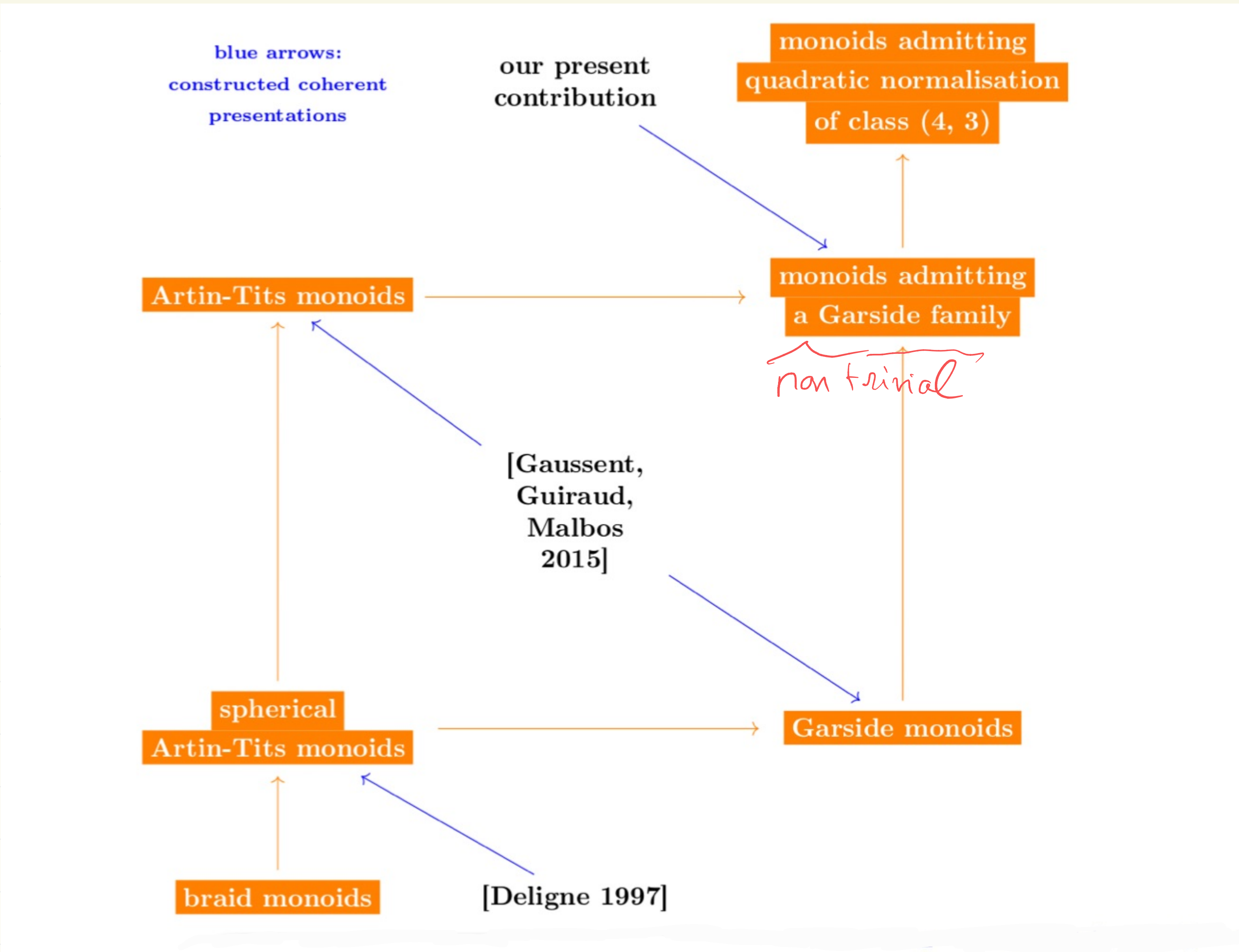


Coherent presentations of monoids (informally)

Presentation $\langle X \mid R \rangle$, where

- X is a set of generators $\rightsquigarrow$ 1-generating cells
- R is a set of pairs $u = v$ (u, v words over X) $\rightsquigarrow$ $u, v \in X^*$

set of generating relations (2-generating cells)

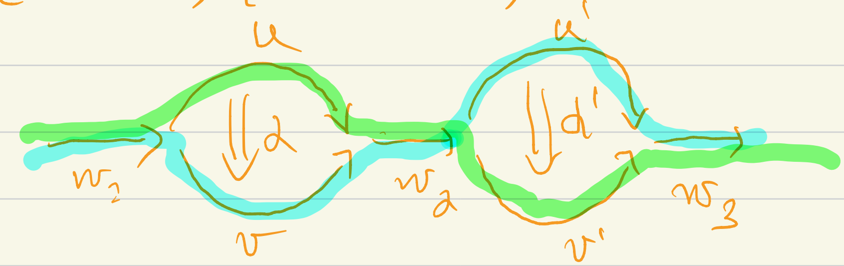
The monoid presented is $\boxed{X^*/\simeq_R}$, where $\simeq_R$ is the smallest congruence containing R, obtained as the transitive closure of steps (for each $u \overset{\alpha}{=} v \in R$)

NOTATION: $t(\alpha) = v$, $s(\alpha) = u$

$$\underbrace{w_1 u w_2 \to w_1 v w_2}_{w_1 \alpha w_2} \quad \text{or} \quad \underbrace{w_1 v w_2 \to w_1 u w_2}_{w_1 \alpha^{-1} w_2}$$

We call relation a formal sequence of such steps (composable), up to

- cancellation of $(w_1 \alpha w_2)(w_1 \alpha^{-1} w_2)$
- and naturality:



$$(w_1 \alpha w_2 u' w_3)(w_1 v w_2 \alpha' w_3) = (w_1 u w_2 \alpha' w_3)(w_1 \alpha w_2 v' w_2)$$

$$\boxed{\xi : u \xrightarrow{\alpha_1} u_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} v}$$

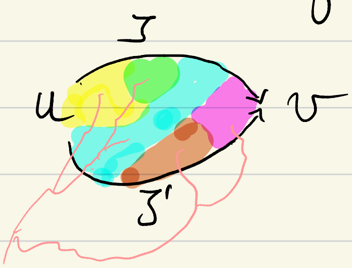
(composable)

Coherent presentations of monoids (informally)

We add a set $\mathbb{R}$ of generating relations among relations $A : \xi_1 \to \xi_2$

3-generating cells

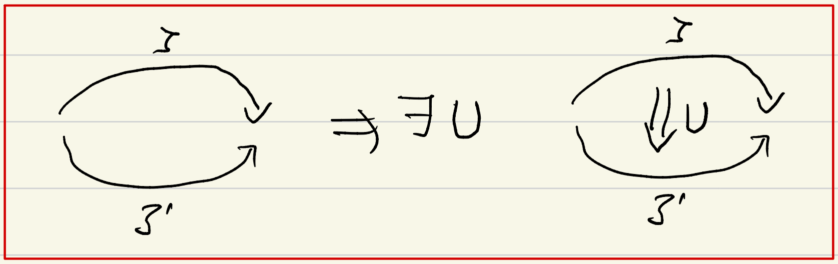
We say that $\langle X \mid R \mid \mathbb{R} \rangle$ is a coherent presentation of $X/_{\equiv R}$ if for every pair (ξ, ξ') of parallel relations



there is a formal composite U of generating relations among relations filling the sphere (ξ, ξ')

relation among relations

all regions = generating relations among relations in context, also called **WHISKERS**

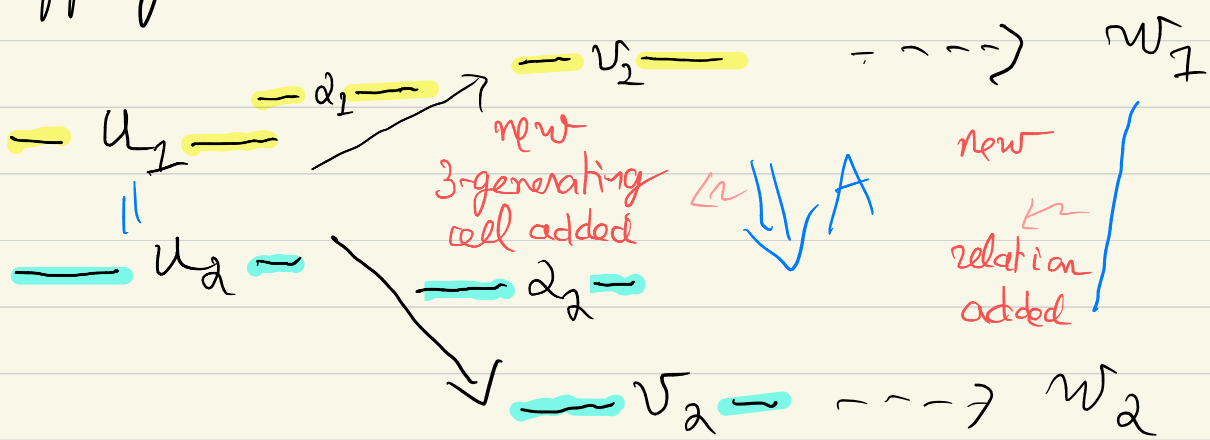


Example: the Klein bottle monoid

$$\langle a, b \mid \underbrace{bab = a}_{\alpha} \rangle$$

The royal way to coherence

- Start from a terminating presentation: no infinite path of steps $-\alpha-$ ($-\alpha^{-1}-$ forbidden)
- Knuth-Bendix completion: check confluence (enough to check this for (minimal) overlapping situations = critical pairs)



If $w_1 \neq w_2$, add $w_1 \to w_2$ or $w_2 \to w_1$ (whichever maintains termination) Iterate

- Squier completion: on the way, add also generating relations among relations ⇝ BINGO! coherent presentation = SQUIER theorem
- Completion-reduction GUIRAUD–MALBOS–MIMRAM

Cerise sur le gâteau: remove some 3-generating and 2-generating cells (justified by Tietze transformations)

All of this preserves presented monoid, coherence resp.

Knuth-Bendix completion (Klein bottle monoid)

Illustration with $\left\langle a, b \,\middle|\, bab \overset{\alpha}{\Rightarrow} a \right\rangle$

Fix a total order on words as follows

- $u < v$ iff $|u| < |v|$ or $\left(|u| = |v| \text{ and } u <_{lex} v\right)$

length

lexicographic order (on words of equal length) generated from $a < b$

(for instance, $b < aa < ab$) Note $t(\alpha) < s(\alpha)$.

- check critical pairs:

$$\begin{tikzcd}
\overline{bab}ab \arrow[rr, "\alpha ab"] \arrow[dr, "ba\alpha"'] \arrow[r, phantom, "\Downarrow A"] & & aab \\
& baa \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Not confluent ↳ close and fill triangle

orientation chosen to maintain $t(\gamma) < s(\gamma)$ for all generating relations

(→ termination)

- β creates a new critical pair (confluent)

$$\begin{tikzcd}
babaa \arrow[rrr, Rightarrow, bend left, "\alpha aa"] \arrow[dr, Rightarrow, "ba\beta"'] & & & aaa \\
& baaab \arrow[r, Rightarrow, "\beta ab"'] \arrow[r, phantom, shift left=6, "B \Uparrow"] & aabab \arrow[ur, Rightarrow, "aa\alpha"'] &
\end{tikzcd}$$

SQUIER completion completed!

Getting rid of B

Critical triple: $bababab$

$$\begin{tikzcd}
 & aabab \arrow[dr, "aa\alpha"] & \\
bababab \arrow[ur, "\alpha bab"] \arrow[r, "ba\alpha ab"] \arrow[dr, "baba\alpha"'] & baaab \arrow[u, "\beta ab"] \arrow[ul, phantom, "Aab \Uparrow"] \arrow[r, phantom, "\Downarrow B"] & aaa \\
 & babaa \arrow[u, "ba\beta"'] \arrow[ur, "\alpha aa"'] \arrow[ul, phantom, "baA \Uparrow"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & aabab \arrow[dr, "aa\alpha"] & \\
bababab \arrow[ur, "\alpha bab"] \arrow[dr, "baba\alpha"'] \arrow[rr, phantom, "="] & & aaa \\
 & babaa \arrow[ur, "\alpha aa"'] &
\end{tikzcd}$$

NATURALITY

Since we do not consider relations among relations among relations, we get

$$(aa\alpha)(Aab) \circ ((aa\alpha \circ \beta ab)(baA) \circ (B^{-1}(baba\alpha)) = id$$

$$B(baba\alpha) = (aa\alpha)(Aab) \circ ((aa\alpha \circ \beta ab)(baA)$$

$$\boxed{B = ((aa\alpha)(Aab) \circ ((aa\alpha \circ \beta ab)(baA))(baba\alpha^{-1})}$$

Getting rid of A and B

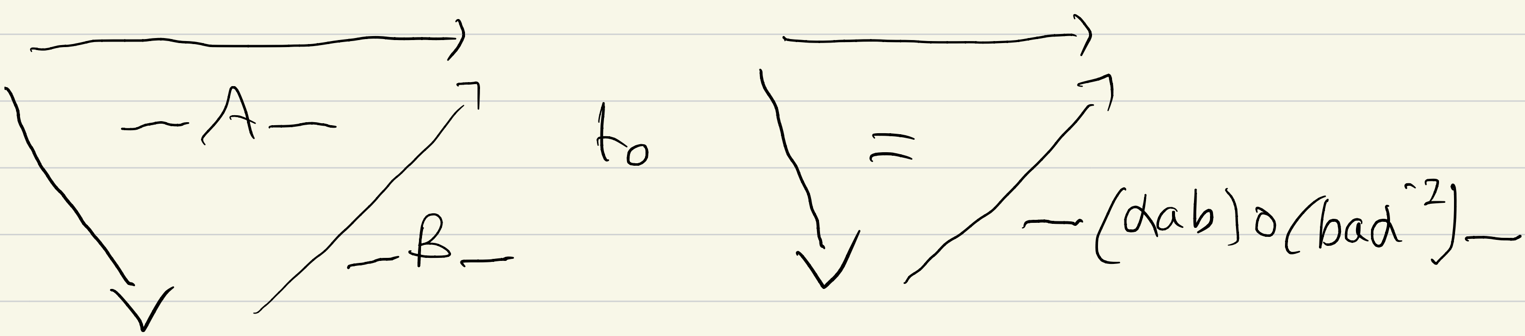
$$\begin{tikzcd} babab \arrow[rr, "\alpha ab"] \arrow[dr, "ba\alpha"'] & \arrow[d, phantom, "A \Uparrow"] & aab \\ & baa \arrow[ur, "\beta"'] & \end{tikzcd}$$

cf. discrete MORSE theory/

BROWN

We can define $\beta := (\alpha ab) \circ (ba\alpha^{-1})$

And then expanding the definition of β in any coherence diagram allows to reduce all occurrences of



In summary:

$$\left.\begin{array}{l} \langle a, b \mid \alpha, \beta \mid A, B \rangle \\ \langle a, b \mid \alpha, \beta \mid A \rangle \\ \boxed{\langle a, b \mid \alpha \mid \emptyset \rangle} \end{array}\right\}$$ are coherent presentations of the Klein bottle monoid.

Coherent presentations of categories (formally)

- monoids are one-object categories
- Polygraphs (or computads) STREET BURRONI:

P_i $(0 \leq i \leq 3)$ set of i-generators together with

free category over $\langle P_0 | P_1 \rangle$ $\simeq$ P_1^*

free strict (2,1) category over $\langle P_0 | P_1 | P_2 \rangle$ $\simeq$ strict 2-category in which all 2-morphisms are invertible

$$\begin{tikzcd}
 & P_1^* & & P_2^\top \arrow[ll, shift left] \arrow[ll, shift right] & & \\
P_0 & P_1 \arrow[l, shift right, "s"'] \arrow[l, shift left, "t"] \arrow[u] & & P_2 \arrow[ull, shift left, "t"'] \arrow[ull, shift right, "s"] \arrow[u] & & P_3 \arrow[ull, shift left, "t"'] \arrow[ull, shift right, "s"]
\end{tikzcd}$$

$$\underbrace{\langle \underbrace{P_0 | P_1}_{\text{1-polygraph}} | P_2 | P_3 \rangle}_{(3,1)\text{-polygraph}}$$

$\langle P_0 | P_1 | P_2 \rangle$: 2-polygraph presenting the category $P_1^* / \simeq_{P_2}$

Definition We say that $\langle P_0 | P_1 | P_2 | P_3 \rangle$ is a $\boxed{\text{coherent presentation}}$ of $P_1^* / \simeq_{P_2}$ (or that P_3 is an acyclic extension of $P_2^\top$) if for every $\zeta, \zeta' \in P_2^\top$ s.t.

$s(\zeta) = s(\zeta')$ and $t(\zeta) = t(\zeta')$, $\exists U \in P_3^\top$ s.t. $s(U) = \zeta$ and $t(U) = \zeta'$

$P_3^\top$ $\simeq$ free strict (3,1) category over $\langle P_0 | P_1 | P_2 \rangle$ $\simeq$ i-morphisms invertible for $i \geq 2$

$$\boxed{\begin{tikzcd} \cdot \arrow[r, bend left=60, "\zeta"] \arrow[r, bend right=60, "\zeta'"'] & \cdot \end{tikzcd} \Rightarrow \exists U \quad \begin{tikzcd} \cdot \arrow[r, bend left=60, "\zeta"] \arrow[r, bend right=60, "\zeta'"'] \arrow[r, phantom, "\Downarrow U"] & \cdot \end{tikzcd}}$$

Homotopical completion-reduction

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\textbf{coherent, reduced}\\ \textbf{(3,1)-polygraph}\end{array}}
& & \boxed{\begin{array}{c}\textbf{coherent, convergent}\\ \textbf{(3,1)-polygraph}\end{array}} \arrow[ll, "\begin{array}{c}\textbf{homotopical}\\ \textbf{reduction}\end{array}"'] \\
\boxed{\begin{array}{c}\textbf{terminating}\\ \textbf{2-polygraph}\end{array}} \arrow[u, "\begin{array}{r}\textbf{homotopical}\\ \textbf{completion-reduction}\end{array}"] \arrow[urr, "\textbf{homotopical completion}" description]
& & \boxed{\begin{array}{c}\textbf{convergent}\\ \textbf{2-polygraph}\end{array}} \arrow[u, "\begin{array}{l}\textbf{Squier}\\ \textbf{completion}\end{array}"']
\arrow[from=2-1, to=2-3, "\begin{array}{c}\textbf{Knuth-Bendix}\\ \textbf{completion}\end{array}"]
\end{tikzcd}
$$

Motivation for coherent presentations

Closely related to

- cofibrant approximations in the canonical model structure on 2-categories, given by Lack
- weak actions of Artin-Tits monoids upon categories, investigated by Deligne

Form

- the first dimensions of polygraphic resolutions of monoids, defined by Métayer, from which abelian resolutions can be deduced

Theorem (Gaussent, Guiraud, Malbos 2015)

Let X be an extended presentation of a category $\mathscr{C}$. TFAE:

- *X is a coherent presentation of $\mathscr{C}$;*
- *$\widetilde{X}$ is a cofibrant approximation of $\mathscr{C}$ (viewed as a 2-category);*
- *for every 2-category $\mathscr{D}$, the category of 2-functors from $\widetilde{X}$ to $\mathscr{D}$ and the category of pseudofunctors from $\mathscr{C}$ to $\mathscr{D}$ are equivalent, and this equivalence is natural in $\mathscr{D}$.*

Overview of relevant classes of monoids

monoids admitting
quadratic normalisation
of class $(4, 3)$
[Dehornoy, Guiraud 2016]

[Dehornoy, Dyer, Hohlweg 2015]

monoids admitting
a Garside family
[Dehornoy et al. around 2010]

Digne, Godelle, Krammer, Michel

Artin-Tits monoids

[Michel 1999]

spherical
Artin-Tits monoids

Garside monoids
[Dehornoy, Paris 1999]

[Brieskorn, Saito 1972]

[Deligne 1972]

braid monoids
[Garside 1969]

Coxeter group: group W presented by

$$\left\langle S \text{ finite} \,\middle|\, \left\{ s^2 = 1, sts\cdots = tst\cdots \,\middle|\, s,t \in S \right\} \right\rangle$$

Spherical Artin-Tits monoid corresponding to a finite Coxeter group W:

$$B^+(W) = \langle S \text{ finite} \,|\, \{ sts\cdots = tst\cdots \,|\, s,t \in S \} \rangle^+$$

Examples

The permutation group S_n, e.g. $S_3 = \left\langle s,t \,\middle|\, s^2 = t^2 = 1, tst = sts \right\rangle$

The braid monoid $B_n^+ = B^+(S_n)$, e.g. $B_3 = \langle s,t \,|\, tst = sts \rangle^+$

Some properties of Artin-Tits monoids

- cancellative
- contain no nontrivial invertible element
- admit conditional right-lcms
- noetherian

Garside's presentation of Artin-Tits monoids

- Introduced by Deligne (1997) for spherical Artin-Tits monoids, and by Michel (1999) for general Artin-Tits monoids

Graphical notation

- $\boxed{\widehat{u \quad v}}$ for $u, v \in B^+(W)$: $\ell(uv) = \ell(u) + \ell(v)$ holds in W
- generalised to a greater number of elements

Definition

Garside's presentation of $B^+(W)$ is a 2-polygraph $\mathrm{Gar}_2(W)$ having:

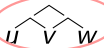
- a single generating 0-cell,
- elements of $W \setminus \{1\}$ as generating 1-cells,
- and a generating 2-cell $\boxed{\mathrm{Gar}_2(W) = \langle * \mid W\setminus\{1\} \mid \{\alpha_{u,v} \mid u \hat{\ } v\} \rangle}$

$$\alpha_{u,v} : u|v \Rightarrow uv$$

for all $u, v \in W \setminus \{1\}$ such that $\widehat{u \quad v}$.

Garside's coherent presentation of Artin-Tits monoids

$\mathrm{Gar}_3(W)$: the extended presentation of $B^+(W)$ obtained by adjoining to $\mathrm{Gar}_2(W)$ a 3-cell $A_{u,v,w}$ for all $u, v, w \in W \setminus \{1\}$

s.t. 

$$\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "A_{u,v,w}"] & & uvw \\
 & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}$$

Theorem (Gaussent, Guiraud, Malbos 2015)

For every Coxeter group W, the Artin-Tits monoid $B^+(W)$ admits $\mathrm{Gar}_3(W)$ as a coherent presentation.

means $u \frown v$, $v \frown w$ and $\ell(uvw) = \ell(u) + \ell(v) + \ell(w)$

Garside monoids

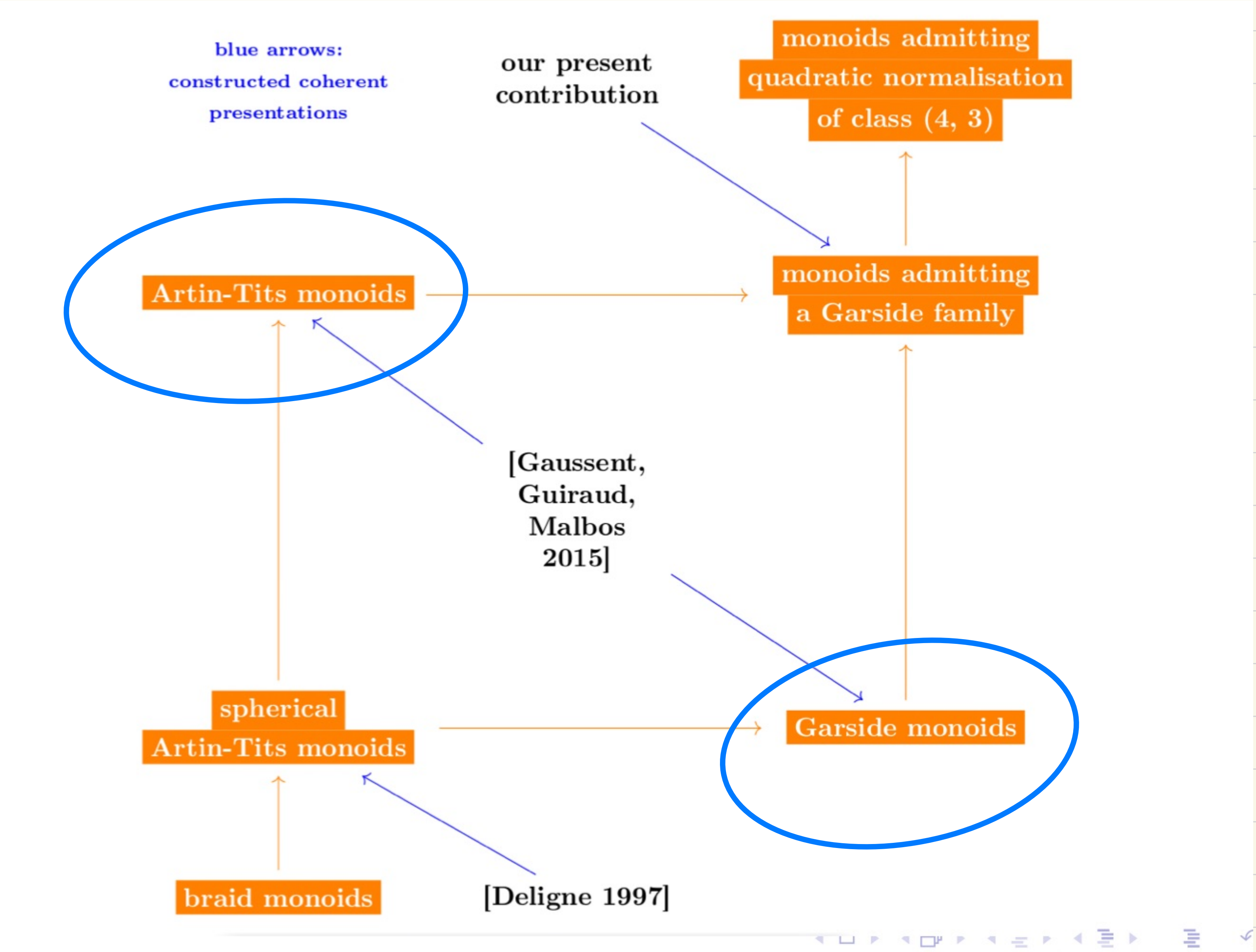
Definition

A Garside monoid is a pair (M, Δ) such that the following conditions hold:

- M is a cancellative monoid;
- there is a map $\lambda : M \to \mathbb{N}$ such that $\lambda(fg) \geq \lambda(f) + \lambda(g)$ and $\lambda(f) = 0 \implies f = 1$;
- every two elements have a left-gcd and a right-gcd and a left-lcm and a right-lcm;
- $\boxed{\Delta \in M\text{, called the Garside element, is such that the left and the right divisors of } \Delta \text{ coincide, and they generate } M\text{;}}$
- the family of all divisors of Δ is finite.

Theorem (Gaussent, Guiraud, Malbos 2015)

Every Garside monoid M admits $\mathrm{Gar}_3(M)$, with $u \frown v$ denoting $uv \in \mathrm{Div}(\Delta)$, as a coherent presentation.


blue arrows:
constructed coherent
presentations
our present
contribution
monoids admitting
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Artin-Tits monoids
monoids admitting
a Garside family
[Gaussent,
Guiraud,
Malbos
2015]
spherical
Artin-Tits monoids
Garside monoids
braid monoids
[Deligne 1997]

Definition of Garside family

Let S be a subfamily of a left-cancellative monoid M

- Greedy decomposition: an S-word $g_1|\cdots|g_q$ is said to be S-normal if for all $i < q$

strict if $g_q \neq 1$

$$\forall h \in S, \forall f \in M, (h \preceq fg_ig_{i+1} \implies h \preceq fg_i)$$

$$\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "f"'] & \bullet \arrow[d] \arrow[dr, bend left] & \\
\bullet \arrow[r, "g_i"] & \bullet \arrow[r, "g_{i+1}"] & \bullet
\end{tikzcd}$$

- Garside family in M: a subfamily S such that every element of M admits an S-normal decomposition

M has no nontrivial invertible element $\Rightarrow$ $\boxed{\text{unique strict } S\text{-normal decomposition}}$

Examples of Garside families

- Every Artin-Tits monoid $B^+(W)$ admits a Garside family given by W
- Every Artin-Tits monoid $B^+(W)$ admits a finite Garside family (Dehornoy, Dyer, Hohlweg 2015)
- In the particular case of a braid monoid, the family of all simple braids is a Garside family
- Every Garside monoid (M, Δ) has a finite Garside family given by $\mathrm{Div}(\Delta)$
- The monoid B_∞^+ of all positive braids on infinitely many strands indexed by positive integers admits

$$S_\infty = \bigcup_{n \geq 1} \mathrm{Div}(\Delta_n)$$

as a Garside family

Properties of Garside families

Let M be a left-cancellative monoid having no nontrivial invertible element, and S a Garside family in M

- S is closed under right divisor and right-mcm.
- Normalisation map $N^S : S^* \to S^*$ assigns to each $w \in S^* \setminus \{1\}$ the strict S-normal decomposition of the evaluation of w; and $N^S(1) = 1$.
- N^S is left-weighted, i.e. for all $s, t \in S$, the element s is a left divisor in M of the first letter of $N^S(s|t)$.
- Rewriting rules $s|t \Rightarrow N^S(s|t)$, for all $s, t \in S \setminus \{1\}$ with $s|t$ not S-normal, yield a convergent presentation of M.

U|

- Let M be a left-cancellative monoid containing no nontrivial invertible element, and $S \subseteq M$ a Garside family s.t. $1 \in S$. Then M admits, as a presentation, the 2-polygraph $\mathrm{Gar}_2(S)$, with $u \frown v$ denoting $uv \in S$.

$$\alpha_{u,v} : \underbrace{u|v \Rightarrow uv}_{u\frown v}$$

$\boxed{\mathrm{Gar}_2(S) = \langle * \mid S\setminus\{1\} \mid \{\alpha_{u,v} \mid u \frown v\} \rangle}$

Our main theorem

Definition

A subfamily S of a monoid M is locally right-noetherian if there exists no $g \in S$ admitting an infinite sequence $(h_n)_{n=1}^{\infty}$ in $S \cap \mathrm{Div}(g)$ such that for every n there exists a non-invertible f_n in S satisfying $h_n f_n = h_{n+1}$.

Theorem

Let M be a left-cancellative monoid containing no nontrivial invertible element, and $S \subseteq M$ is a locally right-noetherian Garside family containing 1. If M admits right-mcms, then M admits the $(3,1)$-polygraph $\mathrm{Gar}_3(S)$ as a coherent presentation.

$(3,1)$-polygraph $\mathrm{Gar}_3(S)$

- $\mathrm{Gar}_2(S)$
- and 3-cell $A_{u,v,w}$ for all $u, v, w \in S \setminus \{1\}$ s.t.

$uv \in S$, $vw \in S$, $uvw \in S$

$$
\begin{tikzcd}
& uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "A_{u,v,w}"] & & uvw \\
& u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}
$$

$\underline{\mathrm{Gar}_3}(S) := \underline{\mathrm{Gar}_2}(S)$ + generating 3-cells

$$
\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "A_{u,v,w}"] & & uvw \\
 & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
u|v|w \arrow[rr, Rightarrow, bend left, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & B_{u,v,w} & uv|w \\
 & u|vw \arrow[ur, Rightarrow, "\beta_{u,v,w}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & uv|wx \arrow[dr, Rightarrow, "\beta_{uv,w,x}"] & \\
u|v|wx \arrow[ur, Rightarrow, "\alpha_{u,v}|wx"] \arrow[dr, Rightarrow, "u|\beta_{v,w,x}"'] \arrow[rr, phantom, "C_{u,v,w,x}"] & & uvw|x \\
 & u|vw|x \arrow[ur, Rightarrow, "\alpha_{u,vw}|x"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
u|v|wx \arrow[rrr, Rightarrow, bend left, "\alpha_{u,v}|wx"] \arrow[dr, Rightarrow, "u|\beta_{v,w,x}"'] & & & uv|wx \\
 & u|vw|x \arrow[r, Rightarrow, "\beta_{u,v,w}|x"'] \arrow[r, phantom, shift left=6, "D_{u,v,w,x}"] & uv|w|x \arrow[ur, Rightarrow, "uv|\alpha_{w,x}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & uv|w|x \arrow[dr, Rightarrow, "uv|\alpha_{w,x}"] & \\
u|vw|x \arrow[ur, Rightarrow, "\beta_{u,v,w}|x"] \arrow[dr, Rightarrow, "u|\alpha_{vw,x}"'] \arrow[rr, phantom, "E_{u,v,w,x}"] & & uv|wx \\
 & u|vwx \arrow[ur, Rightarrow, "\beta_{u,v,wx}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & uv|w|xy \arrow[drr, Rightarrow, "uv|\alpha_{w,xy}"] & & \\
u|vw|xy \arrow[ur, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[dr, Rightarrow, "u|\beta_{vw,x,y}"'] & & F_{u,v,w,x,y} & uv|wxy \\
 & u|vwx|y \arrow[r, Rightarrow, "\beta_{u,v,wx}|y"'] & uv|wx|y \arrow[ur, Rightarrow, "uv|\alpha_{wx,y}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & uv|w|xy \arrow[dr, Rightarrow, "uv|\beta_{w,x,y}"] & \\
u|vw|xy \arrow[ur, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[dr, Rightarrow, "u|\beta_{vw,x,y}"'] \arrow[rr, phantom, "G_{u,v,w,x,y}"] & & uv|wx|y \\
 & u|vwx|y \arrow[ur, Rightarrow, "\beta_{u,v,wx}|y"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & uv|xy \arrow[dr, Rightarrow, "\beta_{uv,x,y}"] & \\
u|\underbrace{vxy}_{\boxed{v(xy) = (vx)y}} \arrow[ur, Rightarrow, "\beta_{u,v,xy}"] \arrow[rr, Rightarrow, "\beta_{u,vx,y}"'] & H_{u,v,x,y} & uvx|y
\end{tikzcd}
$$

degenerate case

$$
\boxed{
\begin{tikzcd}
 & uv_1|w_1 = uv_1|x_1y \arrow[dr, Rightarrow, "\beta_{uv_1,x_1,y}"] & \\
u|v_1w_1 \arrow[ur, Rightarrow, "\beta_{u,v_1,w_1}"] \arrow[d, equal] & & uv_1x_1|y \arrow[d, equal] \\
u|v_2w_2 \arrow[dr, Rightarrow, "\beta_{u,v_2,w_2}"'] & I_{u,v_1,w_1,v_2,w_2} & uv_2x_2|y \\
 & uv_2|w_2 = uv_2|x_2y \arrow[ur, Rightarrow, "\beta_{uv_2,x_2,y}"'] &
\end{tikzcd}
}
$$

Completion of $\mathrm{Gar}_2(S)$

α and α form two critical pairs:

$$\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "A_{u,v,w}"] & & uvw \\
 & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}$$

converges

$$\begin{tikzcd}
u|v|w \arrow[rr, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & & uv|w \\
 & u|vw &
\end{tikzcd}$$

$\rightsquigarrow$ add β and B

$$\begin{tikzcd}
u|v|w \arrow[rr, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & \arrow[d, phantom, "B_{u,v,w}"] & uv|w \\
 & u|vw \arrow[ur, Rightarrow, "\beta_{u,v,w}"'] &
\end{tikzcd}$$

$$\underline{\mathrm{Gar}}_2(S) := \mathrm{Gar}_2(S) + \text{generating 2-cells } \beta$$

$$\beta_{u,v,w} : u|vw \Rightarrow uv|w, \quad u, v, w \in S \setminus \{1\}, \quad u \quad v \quad w$$

We shall show that $\underline{\mathrm{Gar}}_2(S)$ is

convergent $\nearrow$ confluence

$\searrow$ termination

Confluence of critical pairs

$\boxed{\mathrm{I}}$

The only case which needs attention wrt Guiraud - Gaussent - Malbos is this critical branching:

$$\begin{tikzcd}[column sep=large]
 & uv_1 \mid \underbrace{w_1}_{=\, x_1 y} \arrow[dr, "\beta_3"] & \\
\begin{array}{c} u \mid \underbrace{v_1 w_1} \\ \shortparallel \\ v \\ \shortparallel \\ u \mid \overbrace{v_2 w_2} \end{array} \arrow[ur, "\beta_1"] \arrow[dr, "\beta_2"'] & & \begin{array}{c} \underbrace{uv_1x_1} \mid y \\ \shortparallel \\ \text{r-mcm}(v_1,v_2) \\ \shortparallel \\ \overbrace{uv_2x_2} \mid y \end{array} \\
 & uv_2 \mid \underbrace{w_2}_{=\, x_2 y} \arrow[ur, "\beta_4"'] &
\end{tikzcd}$$

where

$$\begin{tikzcd}
 & v \arrow[d, dash, "y"] & \\
 & \text{r-mcm}(v_1,v_2) \arrow[dl, dash, "x_1"'] \arrow[dr, dash, "x_2"] & \\
v_1 & & v_2
\end{tikzcd}$$

$\bullet_1$ Applicability of $\beta_1, \beta_2 \Rightarrow$ $u, v_1, v_2, w_1, w_2, uv_1, uv_2, v \in S$

$\bullet_2$ $\underbrace{u\,(\text{r-mcm}(v_1,v_2))}_{=\, uv_1x_1 \,\in S} = \underbrace{\text{r-mcm}(uv_1, uv_2)}_{\in S}$ by $\bullet_1$ and $\bullet_1$

$\bullet_3$ $x_1, x_2, y \in S$ by $\bullet_2$ and $\bullet_2$

$\bullet_4$ Applicability of β_3: — underlined stuff + $x_2 y \in S$ by $\bullet_1^{(v)}$ and $\bullet_2$

idem β_4

Properties used: $\bullet_1$ closure of S under right-mcm

$\bullet_2$ closure of S under right division

Termination of $\alpha+\beta$

- α terminates (decreases length of S-words)

  Therefore, we can suppose an infinite reduction with β-steps only.

Notation $\quad \underline{u_1|\cdots}|\underbrace{u_i|vw}|\underline{\cdots|u_p} \to_i \underline{u_1|\cdots}|u_i v|w|\underline{\cdots|u_p}$

- Since β preserves length, all steps are $\to_{i_n}$, with $i_n<p$

  By pigeon hole principle, $\exists j$ s.t. $i_n = j$ infinitely often.

  Take $\boxed{j_0}$ the minimal such j

  We can suppose that all steps are $\to_{i_n}$ with $\boxed{i_n \geqslant j_0}$

- Consider, say

$$
\begin{array}{lll}
 & \text{———}\ \overbrace{|u|vw|\ \text{———}}^{U_0} & \\
\to_{j_0} & \text{———}\ \overbrace{|uv|w|\ \text{———}}^{U_1} & \\
(j>j_0+1)\ \to_{j} & \text{———}\ \overbrace{|uv|w|\ \text{———}}^{U_2} & (\text{———} = x_1x_2|\text{———}) \\
\to_{j_0+1} & \text{———}\ \overbrace{|uv|wx_1|x_2\text{———}}^{U_3} & \\
\to_{j_0} & \text{———}\ \overbrace{|uvw|x_1|x_2\text{———}}^{U_4} &
\end{array}
$$

We note that $uv, uvw \in S$ and that uv left divides uvw, i.e.

$\boxed{\text{$j_0$-th letter after a j_0-step left divides j_0-th letter after next j_0-step}}$

- To contradict local right-noetherianity, we are left to find $g \in S$ s.t. $uv, uvw, \cdots$ left divide g. We use the following property

- $_3$ $\boxed{N^S(u|v|w) = N^S(u|N^S(v)|w)}$, which entails

- $N^S(U_1) = \cdots N^S(U_4) = \cdots = U \rightsquigarrow$ Take $g =$ first letter of $N^S(U)$

- (combined with left-weightedness) uv, uvw left divide g

Completing the proof

All the 3-generating cells except A go away

β goes away

$$\begin{tikzcd}[column sep=large, row sep=huge]
\begin{array}{c}\mathrm{Gar}_3(S)\\ \text{coherent, reduced}\end{array} & \begin{array}{c}\underline{\mathrm{Gar}}_3(S)\\ \text{coherent, convergent}\end{array} \arrow[l, "\text{homotopical}"', "\text{reduction}" below] \\
\begin{array}{c}\mathrm{Gar}_2(S)\\ \text{terminating}\end{array} \arrow[u, "\text{homotopical completion-reduction}"] \arrow[r, "\text{Knuth-Bendix}", "\text{completion}"'] & \begin{array}{c}\underline{\mathrm{Gar}}_2(S)\\ \text{convergent}\end{array} \arrow[u, "\text{Squier completion}"']
\end{tikzcd}$$

$$\mathrm{Gar}_3(S) = \mathrm{Gar}_2(S) + \begin{tikzcd}
& uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "A_{u,v,w}"] & & uvw \\
& u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}$$

for all $u, v, w \in S\setminus\{1\}$ s.t. $u \frown v \frown w$

Applications

- Free abelian monoid $\mathbb{N}^{(I)}$ over an infinite basis
  - not of finite type, hence neither Artin-Tits nor Garside
  - Garside family

$$S_I = \left\{ g \in \mathbb{N}^{(I)} \,\middle|\, \forall k \in I, g(k) \in \{0,1\} \right\}$$

  - conditions of Theorem: product on $\mathbb{N}^{(I)}$ is based on the addition of integers
  - $u \frown v$: u and v have disjoint supports

- Monoid B_∞^+ of all positive braids on infinitely many strands indexed by positive integers
  - not of finite type, hence neither Artin-Tits nor Garside
  - Garside family

$$S_\infty = \bigcup_{n \geq 1} \mathrm{Div}(\Delta_n)$$

  - conditions of Theorem: preserved from braid monoids
  - $u \frown v$: uv is a simple braid

Applications, continued

- Dual braid monoid B_n^{+*}
  - generators: $a_{i,j}$ with $1 \leq i < j < n$
  - relations: $a_{i,j} a_{i',j'} = a_{i',j'} a_{i,j}$ for $[i,j]$ and $[i',j']$ disjoint or nested; $a_{i,j} a_{j,k} = a_{j,k} a_{i,k} = a_{i,k} a_{i,j}$ for $1 \leq i < j < k \leq n$
  - Garside monoid: (B_n^{+*}, Δ_n^*) with $\Delta_n^* = a_{1,2} \cdots a_{n-1,n}$
  - further homotopical reduction after Theorem for B_4^{+*}
- Artin-Tits monoid of type $\widetilde{A}_2$
  - presented by

$$\langle \underbrace{\sigma_1, \sigma_2, \sigma_3}_{P_1} \mid \underbrace{\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_3\sigma_1\sigma_3 = \sigma_1\sigma_3\sigma_1}_{P_2} \rangle^+$$

  - Garside family: sixteen right divisors of the elements $\sigma_3\sigma_1\sigma_2\sigma_1$, $\sigma_1\sigma_2\sigma_3\sigma_2$, and $\sigma_2\sigma_3\sigma_1\sigma_3$ (Dehornoy, Dyer, Hohlweg 2015)
  - Theorem and further homotopical reduction:

$$\boxed{\langle * \mid P_1 \mid P_2 \mid \emptyset \rangle}$$

is a coherent presentation