Parabolic closures in some Garside groups

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With Ivan Marin (U. Picardie Jules Verne)

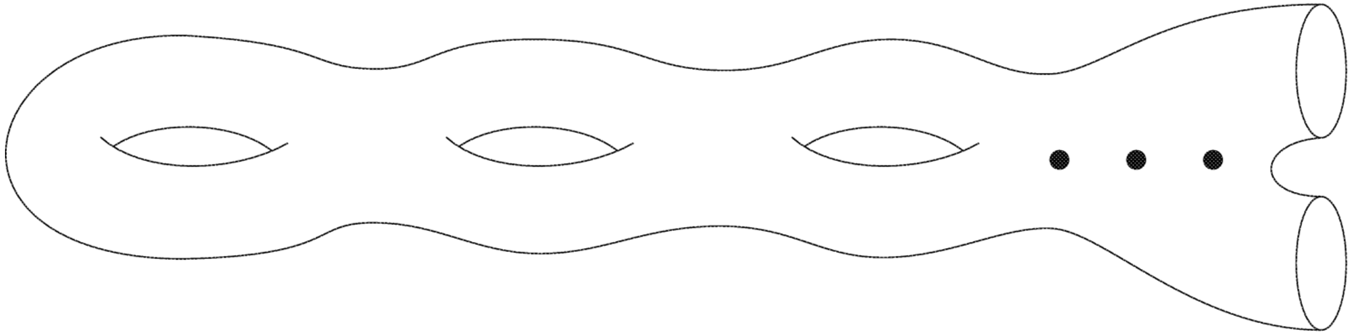
Braids and beyond / Tresses et perspectives

Conference in memory of **Patrick Dehornoy**

Caen, 8-10 September 2021

Mapping class group

Surface S



Mapping class group $$\mathcal{M}(S, \partial S) = \pi_0(\mathrm{Homeo}^+(S, \partial S))$$

To study mapping classes, look at their action on
curves in the surface.

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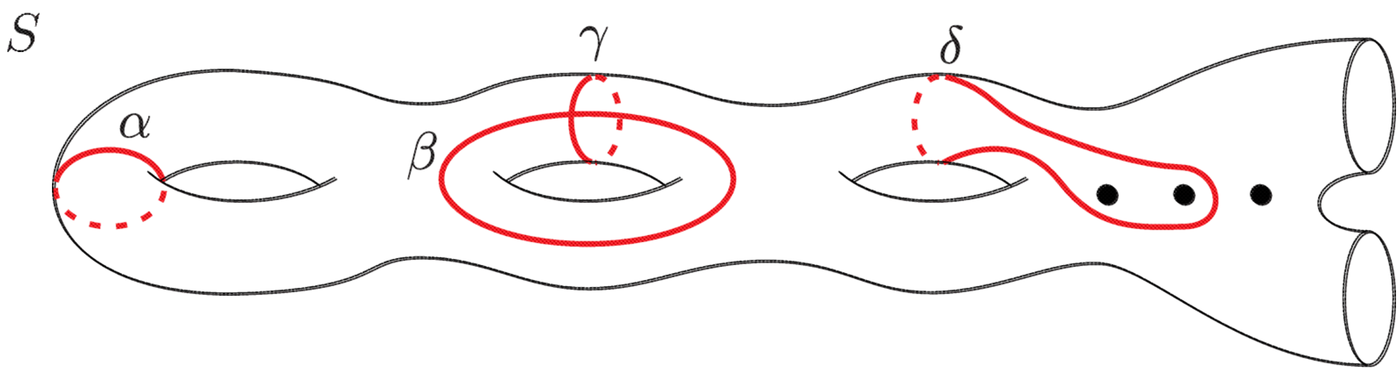
Complex of curves



Curve = isotopy class of non-degenerate simple closed curves

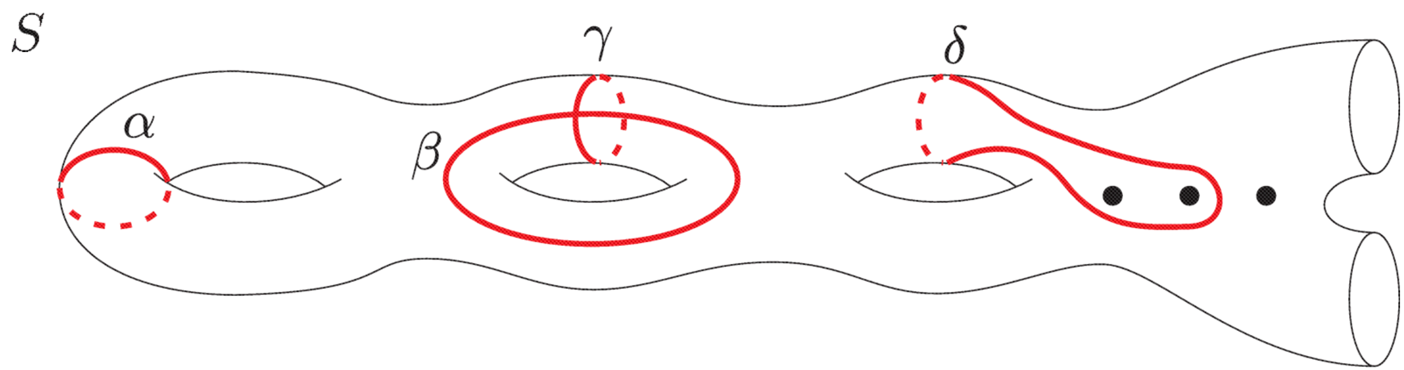
Complex of curves $\mathcal{C}(S)$

Simplicial complex

d-simplex: $\{c_0, c_1, \ldots, c_d\}$ mutually disjoint curves.

$\{\alpha\}$ $\{\alpha, \beta\}$ $\{\alpha, \beta, \delta\}$ $\cancel{\{\alpha, \beta, \gamma\}}$

Complex of curves



Masur-Minsky (1998): $\mathcal{C}(S)$ is δ-hyperbolic.

Aougab (2013)

Clay-Rafi-Schleimer (2013)

Hensel-Przytycki-Webb (2013)

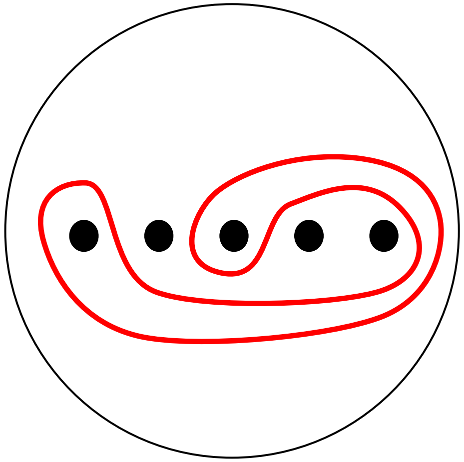
Bowditch (2014)

$\delta \leq 17$

$\mathcal{M}(S, \partial S)$ acts on $\mathcal{C}(S)$ by isometries → Many interesting properties!!!

Braid groups

$\mathbb{D}_n$



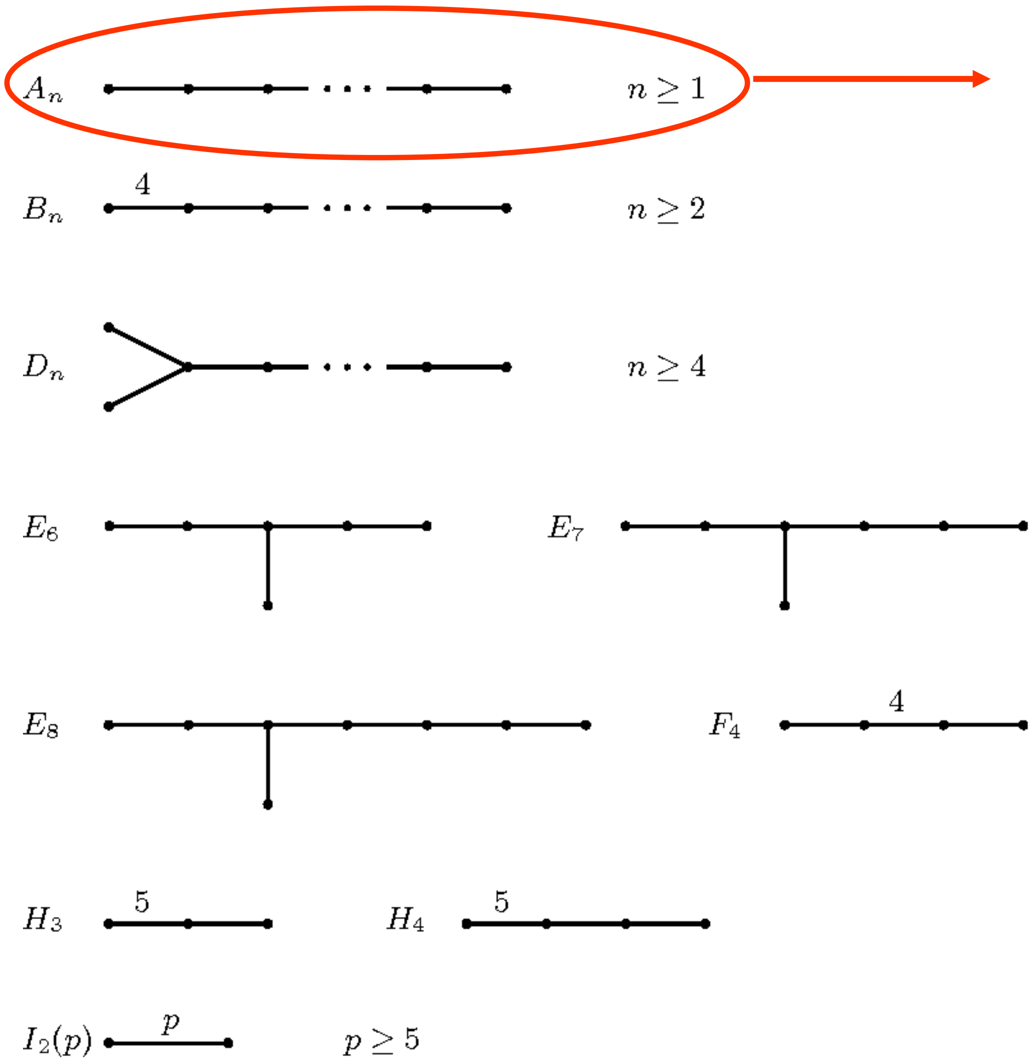
$$\mathcal{M}(\mathbb{D}_n, \partial\mathbb{D}_n) = B_n$$

Braid group on n strands

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & |i-j| \geq 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & |i-j| = 1 \end{array} \right\rangle$$

Algebraic and geometric tools to study braids

Artin-Tits groups of spherical type



Braid groups!

All these are **Garside** groups

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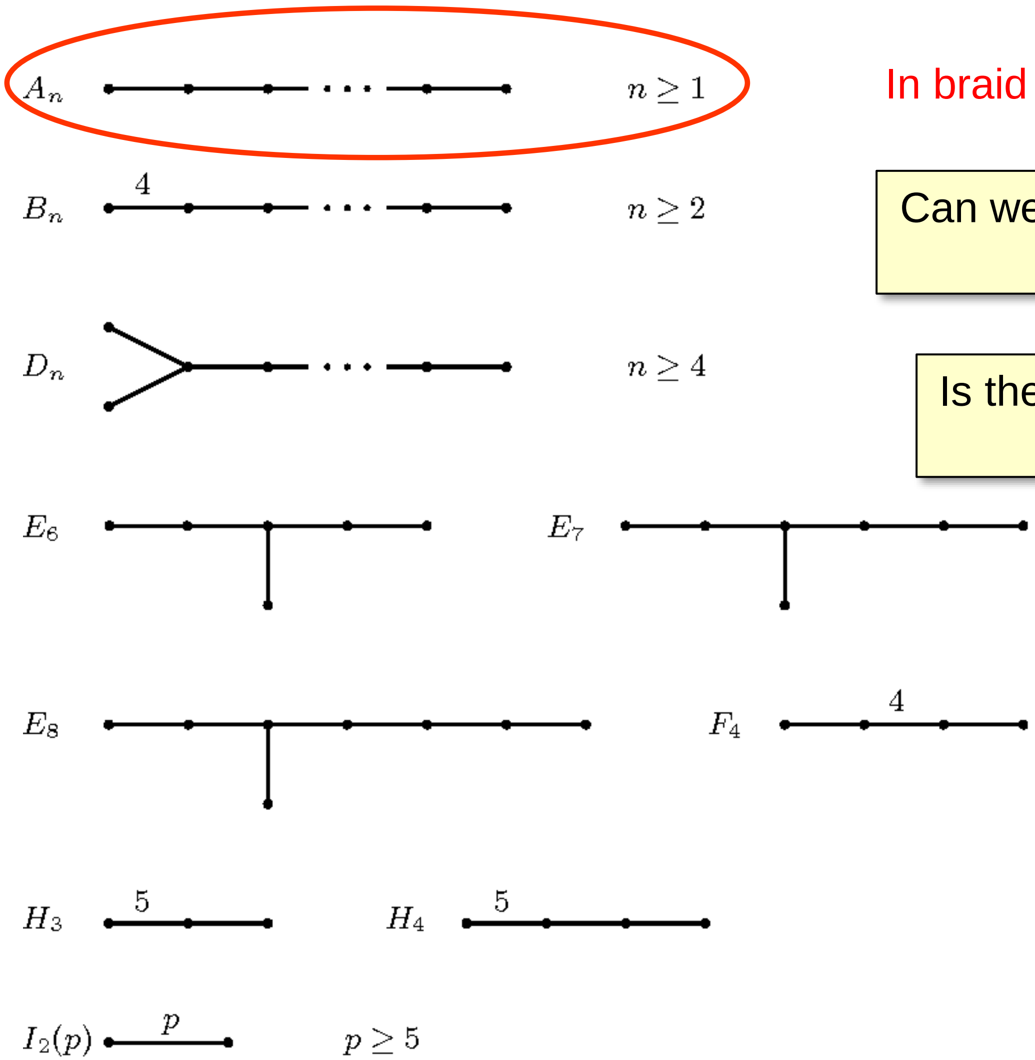
Main questions



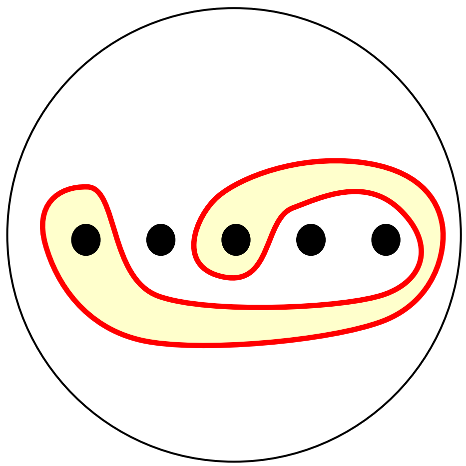
In braid groups we can use geometric tools

Can we extend this to all Artin-Tits groups (of spherical type)?

Is there a natural **algebraic** analogue of the complex of curves?

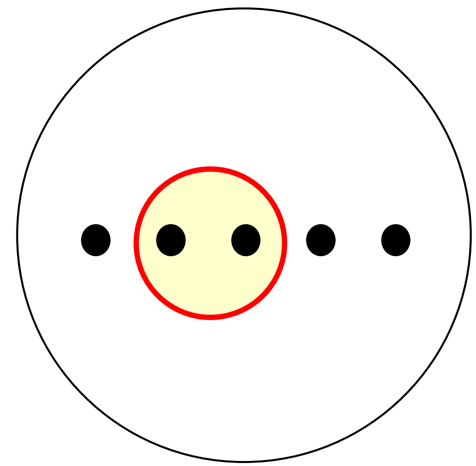
Curves in the disc

$\mathbb{D}_n$



Every curve can be rounded

$\xrightarrow{\alpha}$



Automorphisms with support inside the **curve**

⇕

Conjugate to an irreducible standard parabolic subgroup

⇕

Irred. parabolic subgroup $\alpha A_X \alpha^{-1}$

Automorphisms with support inside the **circle**

⇕

Subgroup generated by some consecutive **standard** generators

⇕

Standard irred. parabolic subgroup A_X

Irreducible parabolic subgroups

Curves in $\mathbb{D}_n$ $\xleftrightarrow{\text{bijection}}$ Proper irreducible parabolic subgroups

Algebraic notion!

In an **Artin-Tits group of spherical type**:

Proper subset of standard generators: $X = \{x_1, \ldots, x_r\} \subset \Sigma$

Standard parabolic subgroup: $A_X = \langle x_1, \ldots, x_r \rangle$ (It is an Artin-Tits group of spherical type) [Van der Lek, 1983]

Parabolic subgroup: $P = \alpha A_X \alpha^{-1}$

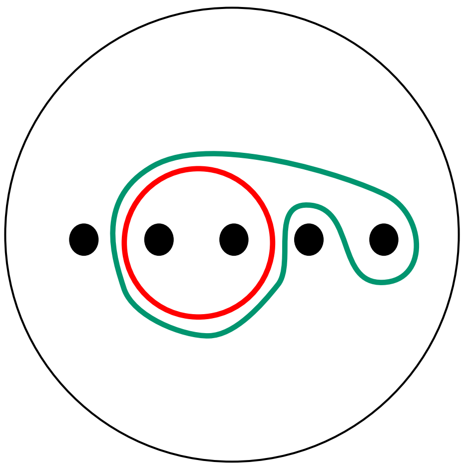
If X is *connected*, P is irreducible

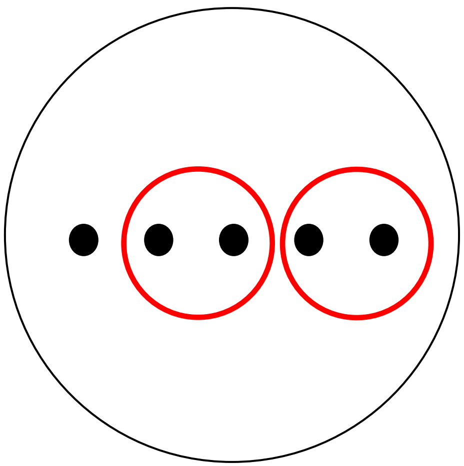
Complex of irreducible parabolic subgroups

Curves in $\mathbb{D}_n$ $\xleftrightarrow{\text{bijection}}$ Proper irreducible parabolic subgroups

Curves are **adjacent** if they are **disjoint**

What does this mean, algebraically?





This happens if and only if their centers commute!

Complex of irreducible parabolic subgroups

$\mathcal{C}(A_\Sigma)$

Simplicial complex

d-simplex: $\{P_0, P_1, \ldots, P_d\}$

Distinct irred. proper parabolic subgroups,
with pairwise commuting centers.

Algebraic definition.

$\mathcal{C}(B_n) \cong \mathcal{C}(\mathbb{D}_n)$ for braid groups.

A_Σ acts on $\mathcal{C}(A_\Sigma)$ by isometries (through conjugation).

Conjecture: Is $\mathcal{C}(A_\Sigma)$ δ-hyperbolic?

Working algebraically

Techniques and properties which are **geometrically** easy…

… can be hard to show **algebraically**.

For instance, **in braid groups**:

The intersection of two parabolic subgroups is a parabolic subgroup

(because the intersection of families of disks is a family of disks)

Hence:

Every element is contained in a minimal parabolic subgroup, called its **parabolic closure**.

(just take the intersection of all parabolic subgroups containing it)

Is this true in all Artin-Tits groups of spherical type?

Is it true in other Garside groups?

Parabolic closure

In A_Σ , **positive elements** = product of standard generators. $\boxed{A_\Sigma^+}$

$u \in A_\Sigma^+ \longrightarrow \boxed{\mathrm{supp}(u) = \{\text{Generators appearing in } u\}}$

(independent of the word representative)

Using **Garside theory**, can show:

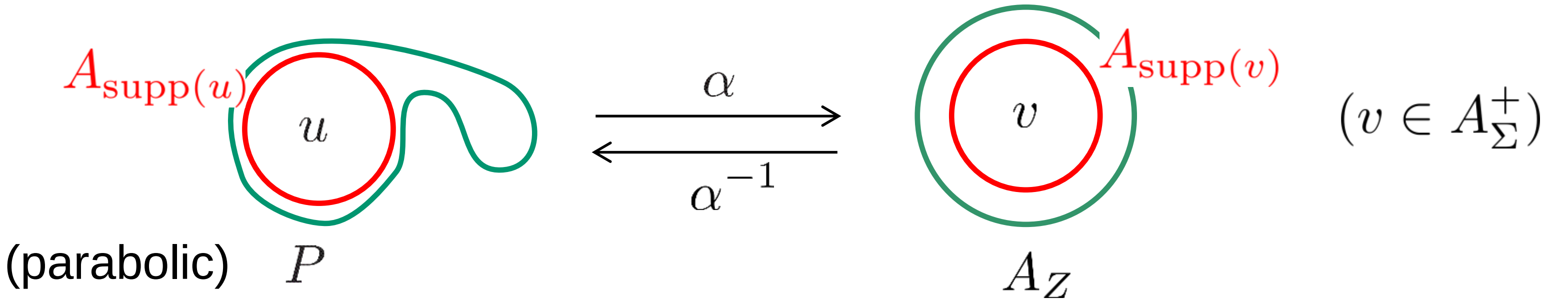
$$\left.\begin{array}{c} u, v \in A_\Sigma^+ \\ \alpha\, u\, \alpha^{-1} = v \end{array}\right\} \Rightarrow \alpha\, A_{\mathrm{supp}(u)}\, \alpha^{-1} = A_{\mathrm{supp}(v)}$$

The support behaves well with conjugations!

$$u \in A_\Sigma^+ \longrightarrow \boxed{\mathrm{supp}(u) = \{\text{Generators appearing in } u\}}$$

(positive element)

$$u \in A_{\mathrm{supp}(u)}$$



$\boxed{A_{\mathrm{supp}(u)} \text{ is the parabolic closure of } u}$

u positive: $u \in A_{\mathrm{supp}(u)}$

x conjugate to positive:

$$\begin{array}{ccc} x & \xrightarrow{\ \alpha\ } & u \\ \cap & & \cap \\ P & \xleftarrow{\ \alpha^{-1}\ } & A_{\mathrm{supp}(u)} \end{array}$$

positive

x any element:

$$\begin{array}{ccc} x & \xrightarrow{\ \alpha\ } & u \\ \cap & & \cap \\ P & \xleftarrow{\ \alpha^{-1}\ } & A_{\mathrm{supp}(u)} \end{array}$$

What conjugate is ok?

How to define its support?

Support of an arbitrary element

Every element in A_Σ can be written in NP-normal form:

$$x = a^{-1}b \qquad a \text{ and } b \text{ positive} \qquad a \wedge b = 1$$

Support of an arbitrary element:

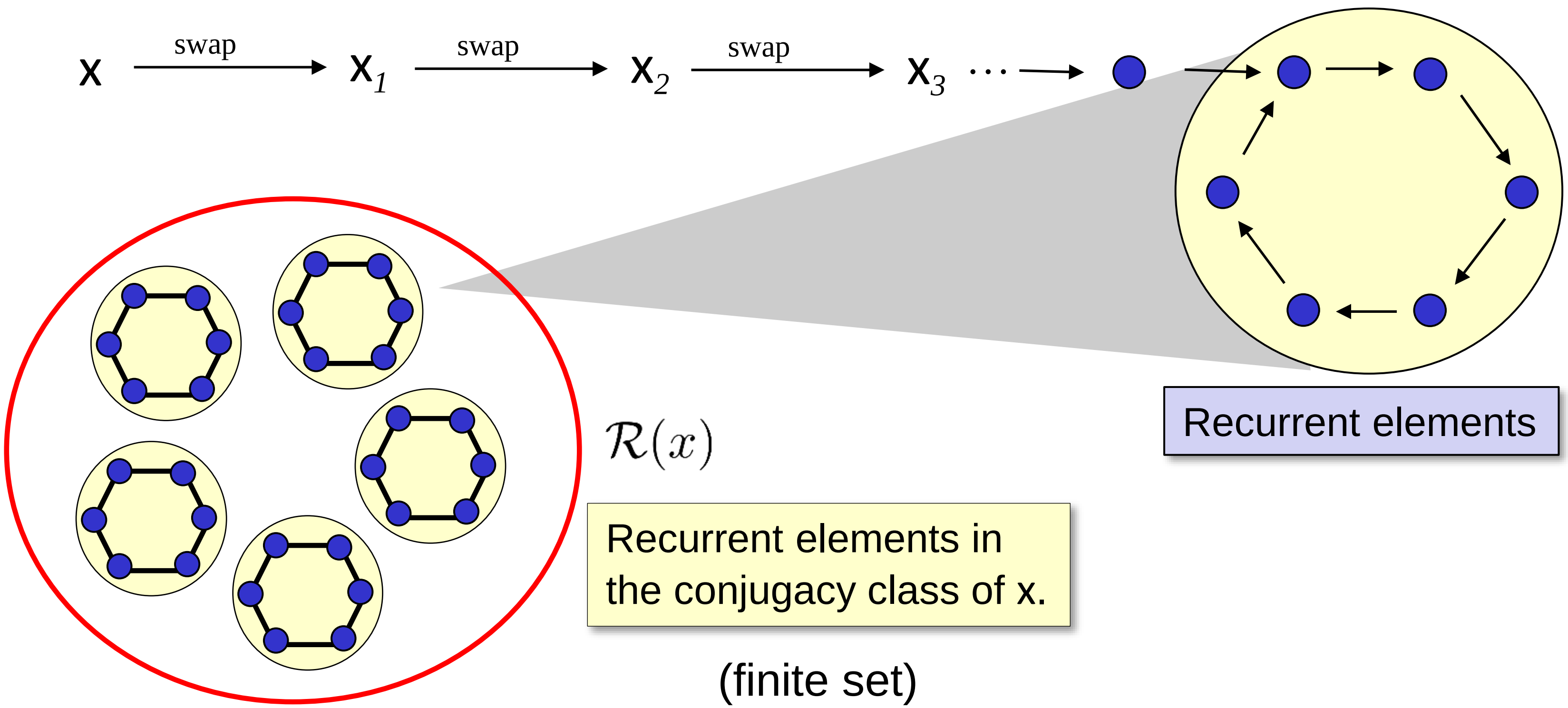
$$\boxed{\operatorname{supp}(x) = \operatorname{supp}(a) \cup \operatorname{supp}(b)}$$

(does not behave well with conjugations)

Recurrent elements

Swap conjugation:

$$a^{-1}b \xrightarrow{\text{swap}} b\,a^{-1} = a_1^{-1}b_1 \xrightarrow{\text{swap}} b_1\,a_1^{-1} = a_2^{-1}b_2 \dashrightarrow$$



Parabolic closure (at last!)

[GM-Marin, 2020]

$$\left.\begin{array}{l} u, v \in \mathcal{R}(x) \\ \alpha\, u\, \alpha^{-1} = v \end{array}\right\} \quad \Rightarrow \quad \alpha\, A_{\mathrm{supp}(u)}\, \alpha^{-1} = A_{\mathrm{supp}(v)}$$

x any element:

$$\begin{array}{ccc} x & \xrightarrow{\quad \alpha \quad} & u \\ \Cap & & \Cap \\ P & \xleftarrow{\quad \alpha^{-1} \quad} & A_{\mathrm{supp}(u)} \end{array}$$

Recurrent

Parabolic closure

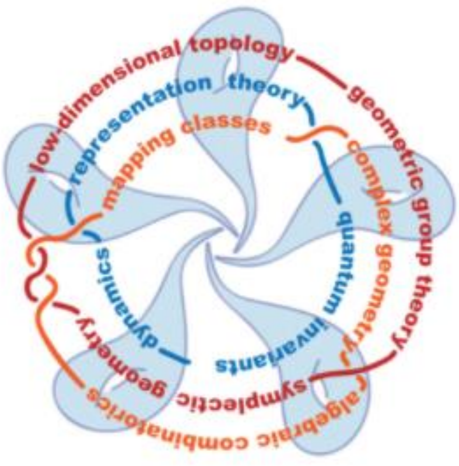
Theorem [CGGW, 2017]
Every element in an Artin-Tits group of spherical type admits a **parabolic closure**.

Commercial break

Institute for Computational and Experimental Research in Mathematics

Providence, RI (USA)

Braids

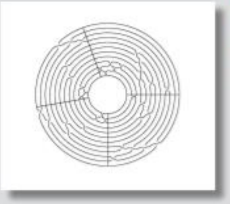


Feb 1 - May 6, 2022

Semester Program

■ FEBRUARY 14 – 18, 2022

Braids in Representation Theory and Algebraic Combinatorics

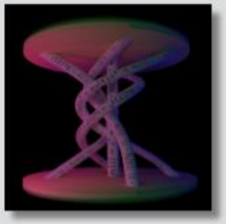


Braid groups and their generalizations play a central role in a number of places in 21st-century mathematics. In modern representation theory, braid groups have come to play an important organizing role, somewhat analogous to the role played by Weyl groups in classical representation theory. Recent advances have established strong connections between homological algebra, geometric representation theory, and algebraic combinatorics. Braid groups appear prominently in many of these connections.

Organizing Committee: **Anna Beliakova,** Universität Zürich; **Ben Elias,** University of Oregon; **Juan González-Meneses,** Universidad de Sevilla; **Anthony Licata,** Australian National University

■ MARCH 21 – 25, 2022

Braids in Symplectic and Algebraic Geometry

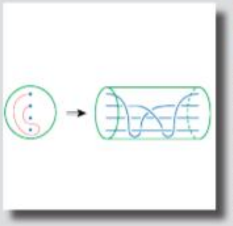


Incarnations of braid groups, or generalizations thereof, naturally arise in a range of active research areas in symplectic and algebraic geometry. This is a rich and diverse ecosystem, and the workshop will aim to bring together speakers from all corners of it. A unifying theme is monodromy. A variety of perspectives lead to a wide array of further geometric applications, from classifications of Stein fillings to the study of spaces of Bridgeland stability conditions.

Organizing Committee: **Inanc Baykur,** University of Massachusetts Amherst; **Anand Deopurkar,** Australian National University; **Benson Farb,** University of Chicago; **Ailsa Keating,** University of Cambridge; **Anthony Licata,** Australian National University

■ APRIL 25 – 29, 2022

Braids in Low-Dimensional Topology



Braids are deeply entwined with low-dimensional topology. Closed braids are knots and links, while viewing braid groups as surface mapping class groups connects the topic to fundamental constructions of three- and four-manifolds. The question of how properties of braids or mapping classes reflect the associated manifolds arises in Dehn surgery, link invariants, and contact and symplectic geometry. The workshop will highlight recent advances in these and other areas of low-dimensional topology where braids and mapping classes play a significant role.

Organizing Committee: **John Etnyre,** Georgia Institute of Technology; **Matthew Hedden,** Michigan State University; **Keiko Kawamuro,** University of Iowa; **Joan Licata,** Australian National University; **Vera Vertesi,** University of Vienna

Garside groups

Can we do the same in other Garside groups?

Problems: Relations are not always homogeneous.

Positive elements can be represented using different letters. (support?)

What is the good notion of parabolic subgroup?

[Godelle, 2004]: Garside group: G Garside element: Δ

Given a **balanced** positive simple element δ $\left(1 \preccurlyeq \delta \preccurlyeq \Delta, \ \ \mathrm{Pref}^+(\delta) = \mathrm{Suff}^+(\delta)\right)$

$G_\delta =$ subgroup generated by the atoms in $\mathrm{Pref}^+(\delta)$

It is called a **standard parabolic subgroup** if $\mathrm{Pref}^+(\delta) = \mathrm{Pref}^+(\Delta) \cap G_\delta^+$

New definitions

Support of positive elements:

$X=\{x_1,\dots,x_r\}$ (atoms) $\Delta_X = x_1\vee\cdots\vee x_r$ (assume is balanced)

Closure: $\overline{X} = \{\text{atoms}\}\cap \mathrm{Pref}^+(\Delta_X)$

Given a positive u: $\boxed{\mathrm{supp}(u) = \overline{\{\text{atoms in a representative of } u\}}}$

Support of general elements: (same)

Swaps and recurrent elements: (same)

Parabolic subgroups: Godelle's definition. $\boxed{A_{\overline{X}} \qquad (\text{for } X\subset\{\text{atoms}\})}$

Parabolic closure

Theorem: [GM-Marin, 2020+] Let G be a Garside group, where lcm's of atoms are balanced.

If support behaves well with conjugations, then every element admits a parabolic closure.

$$\left.\begin{array}{c} u, v \in \mathcal{R}(x) \\ \alpha\, u\, \alpha^{-1} = v \end{array}\right\} \quad \Rightarrow \quad \alpha\, A_{\mathrm{supp}(u)}\, \alpha^{-1} = A_{\mathrm{supp}(v)}$$

Proposition: [GM-Marin, 2020+] Support behaves well with conjugations if and only if

$$\left.\begin{array}{c} u, v \in G^{+} \\ \alpha\, u\, \alpha^{-1} = v \end{array}\right\} \quad \Rightarrow \quad \alpha\, A_{\mathrm{supp}(u)}\, \alpha^{-1} = A_{\mathrm{supp}(v)}$$

Complex braid groups

$W \subset GL_n(\mathbb{C})$ Finite complex reflection group. (generated by pseudo-reflections)

$X = \mathbb{C}^n \setminus \left(\bigcup \mathcal{A} \right)$ (hyperplanes' complement)

$B = \pi_1(X/W)$ (complex braid group)

Natural generalizations of Artin-Tits groups of spherical type

[GM-Marin, 2020+] Topological definition of parabolic subgroups.

Same notion as Godelle's parabolic subgroups

(for those having a Garside structure)

Complex braid groups

Cases:

- Those associated to real reflection groups are spherical Artin-Tits groups.

- $B(r,1,n)$ are spherical Artin-Tits group of type B.

- $B(de,e,n)$ can be related to $B(de,1,n)$.
  $(d>1)$

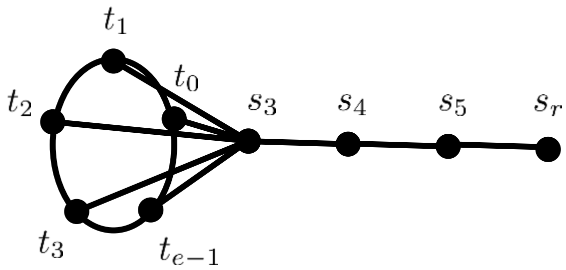
- Shephard groups.

Can use standard monoids

Complex braid groups

Cases:

- $B(e,e,r)$ [Corran-Picantin, 2011] It admits the following Garside monoid:



Braid relations and… $t_1t_0 = t_2t_1 = t_3t_2 = \cdots = t_0t_{e-1}$

Needs Godelles' definition

- Exceptional cases are… exceptional. Use Bessis' monoids in some of them.

Complex braid groups

[GM-Marin, 2020+] Using the mentioned Garside structures, one has:

$$\left.\begin{array}{c} u, v \in \mathcal{R}(x) \\ \alpha\, u\, \alpha^{-1} = v \end{array}\right\} \quad \Rightarrow \quad \alpha\, A_{\mathrm{supp}(u)}\, \alpha^{-1} = A_{\mathrm{supp}(v)}$$

Hence:

Theorem [GM-Marin, 2020+]

Every element in a complex braid group admits a **parabolic closure**.

And **the intersection of parabolic subgroups is parabolic.**

$(G_{31}?)$

MERCI, PATRICK !