Some problems in Artin groups

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Conference in memory of Patrick Dehornoy
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Contents

Definitions

The standard presentation for an **Artin (or Artin-Tits) group** is

$$\langle a_1, \ldots, a_n \mid {}_{m_{ij}}(a_i, a_j) = {}_{m_{ij}}(a_j, a_i) \ \text{ for each } \ i \neq j \rangle$$

where

$$m_{ij} = m_{ji} \in \mathbb{N} \cup \{\infty\}, \quad m_{ii} = 1, \quad m_{ij} \geq 2 \ (i \neq j)$$

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For example, with $n = 3$, $m_{12} = 3$, $m_{23} = 2$, $m_{13} = 5$, we have:

$$\langle a_1, a_2, a_3 \mid a_1a_2a_1 = a_2a_1a_2,\ a_2a_3 = a_3a_2,\ a_1a_3a_1a_3a_1 = a_3a_1a_3a_1a_3 \rangle$$

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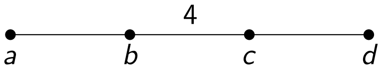
Adding the relations $a_i^2 = 1$ to this presentation defines the associated **Coxeter group**, which is more commonly presented as

$$\langle a_1, \ldots, a_n \mid (a_ia_j)^{m_{ij}} = 1 \quad \text{for each} \quad i, j \rangle$$

The number n of generators is the **rank** of the Artin or Coxeter group.

Definitions (ctd)

We can represent an Artin (or Coxeter) group by its associated **Coxeter Diagram** Γ, such as



which represents the Artin group $A(\Gamma) =$

$$\langle a, b, c, d \mid aba = bab,\ bcbc = cbcb,\ cdc = dcd,\ ac = ca,\ ad = da,\ bd = db \rangle.$$

or the Coxeter group $W(\Gamma)$, which has $a^2 = b^2 = c^2 = d^2 = 1$ as additional relations.

Note that $m_{ij} = 2$ if there is no line drawn between vertices i and j, and $m_{ij} = 3$ if there is an unlabelled line between them.

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It is of type **FC** (flag complex) if, for every subset I of $\{1, 2, \ldots, n\}$ such that $m_{ij} \neq \infty$ for all $i, j \in I$, the Artin group $A(\Gamma_I)$ defined by the subdiagram Γ_I of Γ on the vertices in I is of spherical type.

The natural embedding $A(\Gamma_I) \to A(\Gamma)$ is known to be injective for all Artin groups $A(\Gamma)$ and all subsets I of $\{1, 2, \ldots, m\}$ (**v.d.Lek, 1983; Paris, 1997**).

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It is **right-angled** if $m_{ij} = 2$ or ∞ for all $i \neq j$.

Spherical Artin groups

Spherical Type Artin groups have been studied by Breiskorn, Deligne, Crisp, Paris, Dehormoy, etc.

They are **linear** (Krammer, Bigelow, Cohen, Wales).

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Right-angled Artin groups (also known as **graph groups**) are bi-automatic.

Their word and conjugacy problems are solvable in linear time **Liu, Wrathall, Zeger, 1990; Crisp, Godelle, Wiest, 2009**.

They embed into Coxeter groups (**Hsu and Wise, 1999**), and hence are linear.

Large and extra-large type Artin groups

Appel and Schupp proved in 1983–4 using small cancellation theory that Artin groups of large type have solvable word and conjugacy problems.

Artin groups of (sufficiently) large type are **shortlex automatic** (**Holt, Rees, 2012, 2013**); so their word problems are solvable in quadratic time.

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Following earlier proofs of biautomaticity for groups of extra-large type (**Peifer, 1996**) and other large-type examples (**Brady, McCammond, 2000**), it was recently proved **(Huang and Osajda, 2020)** that all Artin groups of large type are are **systolic**, and therefore biautomatic.

In contrast to those for the shortlex automatic structures, the normal forms for these biautomatic structures are derived from geodesics in the associated systolic complex rather than in the group generators, and they are on larger generating sets, which necessarily include the empty word.

The word problem in Artin groups

As we have seen, the word problem is solvable in Artin groups of spherical type and of sufficiently large type.

This has also been proved for a few other types, including Type FC, affine type, and locally non-spherical. In particular, all Artin groups of rank at most 3 have solvable word problems.

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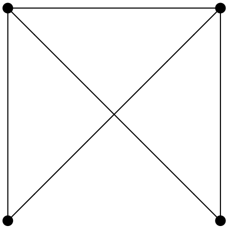
But although we have automaticity for both spherical and large types, the normal forms and the associated theory are different: the theory for spherical types is based on calculations in certain types of monoids with gcd, whereas for large type the arguments are more akin to those in small cancellation theory.

It remains unclear how these ideas could be generalized in a uniform way, and it is unknown whether the word problem is solvable in general Artin groups.

As far as I know, it is unknown whether $A(\Gamma)$ is solvable for the Coxeter diagram below.



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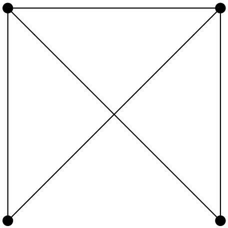

We shall now discuss an attempt by Patrick Dehornoy to address the solvability of the word problem in general Artin groups.

Multifraction reduction

Note that standard presentation of an Artin group G also defines a **monoid** M, and G is the **enveloping group** $\mathcal{U}(M)$ of M.

In general a monoid M need not embed into $\mathcal{U}(M)$, even if M is cancellative, but it was proved by L. Paris in 2001 that $M \to \mathcal{U}(M)$ is an embedding for all Artin monoids M.

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In a sequence of papers, Patrick introduced and developed the theory of **multifraction reduction** for the enveloping groups of certain types of monoid known as **gcd-monoids**. These include the Artin monoids.

Definition

A **multifraction** is a sequence $a_1, a_2, \ldots, a_n$ of elements $a_i \in M$ (so we are working over an alphabet that is usually infinite), which is denoted here by $a_1 / \cdots / a_n$, and which represents the group element $a_1 a_2^{-1} a_3 \cdots a_n^{(-1)^{n-1}}$.

A reduction process is defined consisting of a (usually infinite) set of rewriting rules, each of which replaces a subsequence $a_{i-1}/a_i/a_{i+1}$ of a multifraction by $b_{i-1}/b_i/b_{i+1}$, where the gcd properties of the monoid are used to cancel common left or right divisors, and also to move such divisors to the left.

By convention, if this results in the empty word being the final component of a multifraction, then we delete that component.

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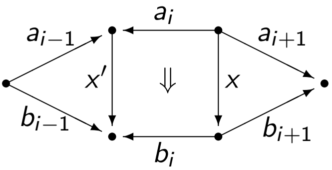
The possible reductions are illustrated in the diagrams below.



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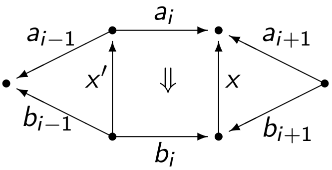

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In Artin monoids of spherical type, any two elements have a (least) common right multiple, and this process reduces every multifraction to one of length at most 2.

It can be regarded as a rewriting system (but with infinitely many letters and rules).

In his first paper, Patrick proved that the process is convergent in an Artin group (equivalently, words representing the same group element reduce to the same irreducible multifraction) if and only if it is of type FC.

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Definition

A rewriting system over the generators of a monoid M is **semi-convergent** if and only if any word that represents the identity element of M can be reduced to the empty multifraction by suitable application of the rewriting rules.

For example the **Dehn algorithm** in a hyperbolic group is semi-convergent but not usually convergent.

In the second of the papers, Patrick made the following ambitious conjecture.

Conjecture A

Multifraction reduction is semi-convergent in all Artin groups.

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He went on to prove the following result, which follows from the non-trivial result that it is decidable whether a given multifraction over an Artin group is reducible or irreducible with the specified rewriting rules.

Proposition (Dehornoy, 2016)

If conjecture A is true, then the word problem is solvable in all Artin groups.

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Proposition (Dehornoy, 2016)

If conjecture A is true, then the word problem is solvable in all Artin groups.

He also formulated some slightly stronger versions of the conjecture that were suitable for computer verification, and he tested them extensively by computer on a wide variety of examples without finding a counterexample.

The conjecture in large-type Artin groups

Can we at least prove this conjecture for Artin groups of (sufficiently) large type?

Not quite, although we cannot disprove it, and we have not found an explicit example in which we are unable to reduce a word representing the identity to the empty word using multifraction reduction.

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One difficulty resulted from the limitations on carrying out the reduction when $i = 1$, when we can only replace a_1/a_2 by b_1/b_2 when a_1 and a_2 have a common right divisor.

We can circumvent this problem by allowing the insertion of an even number of empty components on the left of the input multifraction before we start the reduction process.

In the following definition, we denote the empty word by ϵ, and $(\epsilon/)^k$ denotes the multifraction $\epsilon/\cdots/\epsilon$ of length k.

Definition

A multifraction $a_1/\cdots/a_n$ reduces to $b_1/\cdots/b_m$ using **multifraction reduction with k-padding** if the multifraction $(\epsilon/)^{2k}/a_1/\cdots/a_n$ reduces to $b_1/\cdots/b_m$.

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Theorem (Dehornoy, Holt, Rees, 2018)

Let G be an Artin group of sufficiently large type, and let $a_1/\cdots/a_n$ be a multifraction that represents the identity element of G. Then $a_1/\cdots/a_n$ can be reduced to the empty multifraction using multifraction reduction with k-padding.

In fact we can take $k = 3\ell(\ell+2)/4$, where $\ell = \sum_{i=1}^{n} \ell(a_i)$, and $\ell(a_i)$ is the length of a_i as a word in the generators of the Artin monoid.

Interval groups of Coxeter and quasi-Coxeter elements

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The group $G([1, w])$ is a quotient by a single extra relation of the Artin group $A(\Gamma_w)$ of a **Carter diagram** Γ_w, and $G([1, w])$ turns out to be isomorphic to the Artin group $A(\Gamma)$ for all Coxeter elements w.

Example

$G = W(A_2) = \langle a, b \mid a^2 = b^2 = 1, aba = bab \rangle$, $w = ab$.

We have $w = ab = ca = bc$, where $c = aba = bab$, and

$$G([1, w]) \cong \langle a, b, c \mid ab = ca = bc \rangle \cong \langle a, b \mid aba = bab \rangle = A(A_2).$$

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A **quasi-Coxeter element** $w_0 \in G = W(\Gamma)$ is an irreducible product of n (not necessarily simple) reflections that generate $W(\Gamma)$, and the interval group $G([1, w_0])$ can be defined in the same way.

The element w_0 is called a **proper** quasi-Coxeter element if it is not a Coxeter element. For such elements it is not always true that $G([1, w_0]) \cong A(\Gamma)$.

In recent work, Baumeister, Neaime and Rees have computed presentations of $G([1, w_0])$ for proper quasi-Coxeter elements in the case $\Gamma = D_n$.

There are $\lfloor \frac{n}{2} \rfloor$ conjugacy classes of quasi-Coxeter elements in D_n (including the Coxeter elements), and this is the only infinite family of finite Coxeter groups that have proper quasi-Coxeter elements for all n.

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To state their main result, we need some definitions. First we define the **cyclic commutator (cc)** and **twisted cyclic commutator (tc)** of a sequence of elements $y_1, \ldots, y_n$ in a group.

Definition

$$\begin{aligned} \mathrm{CC}(y_1, y_2, \ldots, y_n) &:= [y_1, w y_n w^{-1}] \text{ where } w = y_2 \cdots y_{n-1} \\ \mathrm{TC}(y_1, y_2, \ldots, y_n)_t &:= [y_1, w y_n w^{-1}] \text{ where } w = y_2^{-1} \cdots y_{t+1}^{-1} y_{t+1} \cdots y_{n-1} \end{aligned}$$

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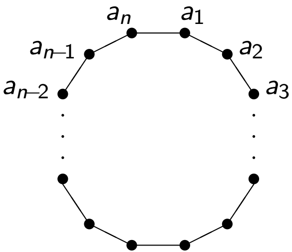
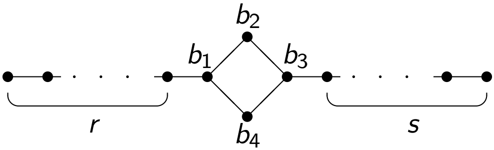
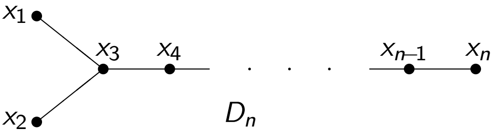
Some Coxeter Diagrams



$\Delta_n = \tilde{A}_{n-1}$



$\Delta_{r,s}$



D_n

Theorem (Baumeister, Neaime, Rees)

Let w_0 be a proper quasi-Coxeter element in $W(D_n)$ for some $n \geq 4$. Then, for some $r, s \geq 0$ with $4 + r + s = n$, we have

$$G([1, w_0)]) \cong A(\Delta_{r,s})/\langle\langle \mathrm{TC}(b_1, b_2, b_3, b_4)_1 \rangle\rangle$$

(*where* $\mathrm{TC}(b_1, b_2, b_3, b_4)_1 = [b_1, b_2^{-1} b_3 b_4 b_3^{-1} b_2]$).

There are $\lfloor \frac{n-2}{2} \rfloor$ possible choices of (r, s) in the above result, each of which corresponds to a unique conjugacy class of quasi-Coxeter elements in $W(D_n)$. So, if we include the Coxeter elements, then we have a total of $\lfloor \frac{n}{2} \rfloor$ classes.

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When w is a Coxeter element, we have

$$\begin{aligned} G([1, w]) \cong A(D_n) &\cong A(\Delta_{r,s})/\langle\langle \mathrm{CC}(b_1, b_2, b_3, b_4) \rangle\rangle \\ &\cong A(\Delta_n)/\langle\langle \mathrm{CC}(a_1, a_2, \ldots, a_n) \rangle\rangle \end{aligned}$$

for all $r, s \geq 0$ with $4 + r + s = n$.

Some further isomorphisms

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Question

Do these groups have solvable word problems?

Proving the isomorphisms

In the result above, we need to prove isomorphisms between two groups defined by finite presentations $G = \langle X \mid R \rangle$ and $H = \langle Y \mid S \rangle$.

Our technique for achieving this can be summarised as follows.

1. Define maps $\phi : X \to H$ and $\psi : Y \to G$.
2. Use the relations S of H to verify that the images under ϕ of the relations R of G are satisfied in H, and similarly for $\psi : H \to G$. This proves that ϕ and ψ extend to homomorphisms $G \to H$ and $H \to G$ respectively.
3. Use the relations R of G to verify that $\psi(\phi(x)) = x$ for for all $x \in X$, and the relations S of H to verify that $\phi(\psi(y)) = y$ for all $y \in X$. This proves that ϕ and ψ are mutually inverse isomorphisms.

But how do we come up with the maps ϕ and ψ in the first step?

We did this essentially by trial and error, starting with the smallest cases. The verifications in the second and third steps can, again in small cases, be checked quickly using the KBMAG package (available as a standalone or from GAP or Magma), which implements the Knuth-Bendix rewriting procedure.

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But if KBMAG fails to verify a relation, then that does not (usually) prove that it does not hold; we may need to run the procedure for longer.

Eventually we succeeded in constructing the isomorphisms in enough small cases to enable us to guess the maps ϕ and ψ in general, and we also succeeded in completing the verifications using inductive proofs without computer assistance.

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