Braid groups, free groups and symplectic Steinberg groups

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Conférence "Tresses et perspectives"
à la mémoire de Patrick Dehornoy,
Université de Caen

Le 8 septembre 2021

Dedication

• I dedicate this lecture to the memory of
our friend and colleague Patrick Dehornoy (1952–2019)

• I have already expressed what Patrick meant to me at his funeral and in the tribute
Souvenirs de Patrick Dehornoy (1952–2019)
(written with Serge Grigorieff, Philippe Toffin, François Digne and Pierre-Louis Curien) which appeared in Gazette des Mathématiciens 164 (2020), 63–69.

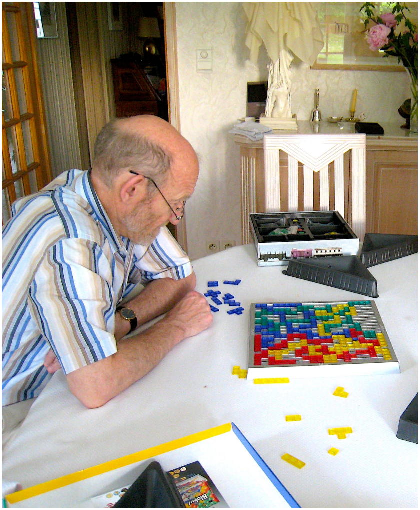
Patrick the Traveller

Patrick and me travelling in Greece (1974)

Patrick the Builder

Patrick working on his house (1982)

Patrick the Player



Patrick absorbed in solving a difficult problem!

Patrick in Montréal

With Christophe Reutenauer in Montréal (2017)

Now back to mathematics...

Synopsis

…Back to the braid groups
that kept Patrick busy over
more than two decades.

- Here is an outline of my talk.
  - I recall how to construct an action of B_{2n+2} on the free group F_{2n}.

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  - Recall what Steinberg groups are and lift $\bar{f}$ to a homomorphism $\widetilde{f} : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$ into the integral Steinberg group of type C_n (this is joint work with François Digne).

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  - Special cases $n = 1, 2$.
  - A few results in the general case (work in progress with François Digne).

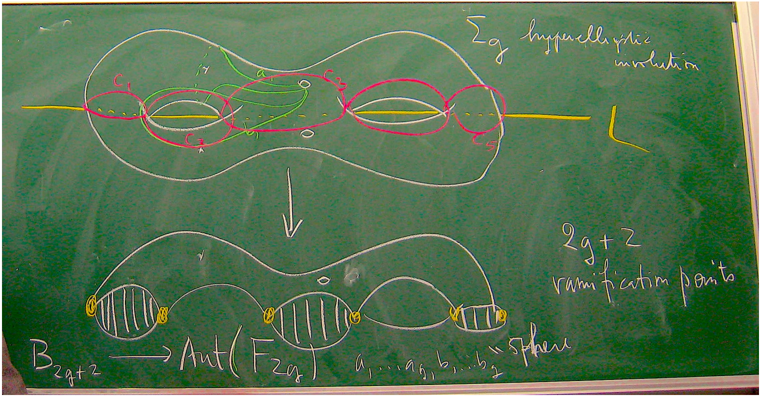
A ramified double covering of a disk

The idea of the braid group B_{2n+2} acting on the free group F_{2n} goes back to work of Magnus & Peluso, Birman (1969)...

• Consider a surface Σ of genus $n \geq 1$, which is invariant under the hyperelliptic involution, which is the reflection in the line L. This line intersects Σ in $2n+2$ points.

• The quotient of Σ by the hyperelliptic involution is a sphere. We thus obtain a double covering $p : \Sigma \to S^2$ of the sphere with $2n+2$ ramification points.

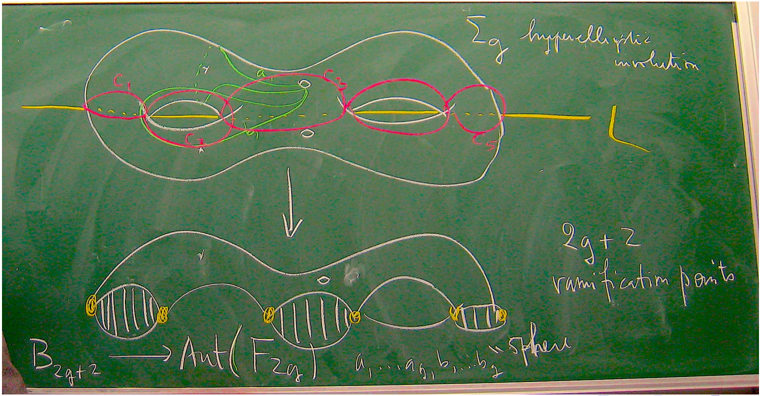
• Removing a small disc ("hole") from the quotient, we obtain a double covering $p : \Sigma_0 \to D$ of a disk D with $2n+2$ ramification points.



An action of the braid group B_{2n+2} on the free group F_{2n}

• The braid group B_{2n+2} can be realized as the mapping class group of the disk D with $2n+2$ distinguished points. Each element of B_{2n+2} can be represented as an orientation-preserving homeomorphism fixing each point of the boundary of D and permuting the distinguished points.

• Lifting each such homeomorphism to a homeomorphism of Σ_0 (fixing the two holes) induces a group homomorphism from B_{2n+2} to the mapping class group of Σ_0, hence to the automorphism group of the fundamental group $\pi_1(\Sigma_0)$. The latter is the free group generated by the loops $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ of the figure.



The homomorphism $B_{2n+2} \to \mathrm{Aut}(F_{2n})$

• Representing each generator $\sigma_1, \ldots, \sigma_{2n+1}$ of B_{2n+2} by a homeomorphism of D, lifting the latter to Σ_0 and computing the action of each lift on the loops $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ yields a homomorphism $f : B_{2n+2} \to \mathrm{Aut}(F_{2n})$.

• We have $f(\sigma_i) = u_i$, where $u_1, \ldots, u_{2n+1}$ are the following automorphisms of the free group $F_{2n} = \langle a_1, \ldots, a_n, b_1, \ldots, b_n \rangle$.

(a) The automorphism u_1 fixes all generators, except b_1 for which

$$u_1(b_1) = a_1 b_1 .$$

(b) The automorphism u_{2n+1} fixes all generators, except b_n for which

$$u_{2n+1}(b_n) = b_n a_n .$$

(c) The automorphism u_{2i} $(1 \leq i \leq n)$ fixes all generators, except a_i for which

$$u_{2i}(a_i) = b_i^{-1} a_i .$$

(d) The automorphism u_{2i+1} $(1 \leq i \leq n-1)$ fixes all generators, except b_i and b_{i+1} for which we have

$$u_{2i+1}(b_i) = b_i a_i a_{i+1}^{-1} \quad \text{and} \quad u_{2i+1}(b_{i+1}) = a_{i+1} a_i^{-1} b_{i+1} .$$

The case $n = 1$ - An exact sequence

• What can we say about the image and the kernel of $f : B_{2n+2} \to \mathrm{Aut}(F_{2n})$ for $n = 1$?

• Together with Christophe Reutenauer (Ann. Mat. Pura Appl. 2007) we showed that the homomorphism $f : B_4 \to \mathrm{Aut}(F_2)$ fits into the exact sequence of groups

$$1 \longrightarrow Z_4 \longrightarrow B_4 \xrightarrow{f} \mathrm{Aut}(F_2) \xrightarrow{\varepsilon} \{\pm 1\} \longrightarrow 1.$$

Here Z_4 is the center of B_4: it is the infinite cyclic group generated by Δ_4^2, where $\Delta_4 = (\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2)\sigma_1$,

• **Conclusion.** The kernel of f is the center Z_4 of B_4 and its image is a subgroup of index 2 of the automorphism group $\mathrm{Aut}(F_2)$.

• What is the homomorphism $\varepsilon : \mathrm{Aut}(F_2) \to \{\pm 1\}$ exactly?

The case $n = 1$ - A diagram of exact sequences

- The map $f : B_4 \to \mathrm{Aut}(F_2)$ fits into the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
1 \to & Z_4 & \longrightarrow & B_4 & \xrightarrow{f} & \mathrm{Aut}(F_2) & \xrightarrow{\varepsilon} & \{\pm 1\} & \to 1 \\
 & \downarrow \cong & & \downarrow q & & \downarrow \mathrm{ab} & & \downarrow \mathrm{id} & \\
1 \to & 2Z_3 & \longrightarrow & B_3 & \xrightarrow{f'} & \mathrm{GL}_2(\mathbb{Z}) & \xrightarrow{\det} & \{\pm 1\} & \to 1
\end{array}$$

Here

- The vertical homomorphism ab is the abelianization map and $\varepsilon = \det \circ \mathrm{ab}$.

- **Keep in mind.** The homomorphism f vanishes on the center of B_4 and the image of $\mathrm{ab} \circ f : B_4 \to \mathrm{GL}_2(\mathbb{Z})$ is $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Ker}(\det)$.

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- The vertical homomorphism $q : B_4 \to B_3$ is defined by $q(\sigma_1) = q(\sigma_3) = \sigma_1$ and $q(\sigma_2) = \sigma_2$. We have $q \circ i = \mathrm{id}$, were $i : B_3 \to B_4$ is the standard inclusion. (Note that there is no retraction of $i : B_n \to B_{n+1}$ when $n > 3$.)

- **Keep in mind.** The homomorphism f vanishes on the center of B_4 and the image of $\mathrm{ab} \circ f : B_4 \to \mathrm{GL}_2(\mathbb{Z})$ is $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Ker}(\det)$.

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- The homomorphism $f' : B_3 \to \mathrm{GL}_2(\mathbb{Z})$ is determined by

$$f'(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f'(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

- **Keep in mind.** The homomorphism f vanishes on the center of B_4 and the image of $\mathrm{ab} \circ f : B_4 \to \mathrm{GL}_2(\mathbb{Z})$ is $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{Ker}(\det)$.

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Symplectic matrices

Back to arbitrary $n \geq 1$. Let us give two (easy to prove) results on the homomorphism

$$f : B_{2n+2} \to \mathrm{Aut}(F_{2n})$$

and its linearization

$$\bar{f} = \mathrm{ab} \circ f : B_{2n+2} \to \mathrm{GL}_{2n}(\mathbb{Z}).$$

- **Proposition.** *For any $n \geq 1$,*
  - *(a) $f(Z_{2n+2}) = 1$, where Z_{2n+2} is the center of B_{2n+2}.*
  - *(b) $\mathrm{Im}\left(\bar{f} : B_{2n+2} \to \mathrm{GL}_{2n}(\mathbb{Z})\right) \subset \mathrm{Sp}_{2n}(\mathbb{Z})$.*

Recall that the symplectic modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$ is the group of matrices $M \in \mathrm{GL}_{2n}(\mathbb{Z})$ such that

$$M^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where M^T is the transpose of M and I_n is the identity matrix of size n.

- **Remarks.**
  - For $n = 1$ we have $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$.

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- **Remarks.**
  - For $n = 1$ we have $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$.
  - It is well known that elements of the mapping class group of a surface Σ preserve the algebraic intersection pairing on $H_1(\Sigma, \mathbb{Z})$, which is a symplectic form.

The case $n = 2$ - Theorem 1

We now consider the special case $n = 2$ for the composite homomorphism

$$\bar{f} = \mathrm{ab} \circ f : B_6 \to \mathrm{Sp}_4(\mathbb{Z}).$$

• In the Special issue of the Journal of Algebra in honor of Patrick Dehornoy (2020) we established the following result.

Theorem 1. *(a) The map $\bar{f} : B_6 \to \mathrm{Sp}_4(\mathbb{Z})$ is surjective.*
(b) Its kernel is the normal subgroup of B_6 generated by the two elements

$$(\sigma_1\sigma_2\sigma_1)^4 \quad \textit{and} \quad (\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5).$$

This yields a braid-type presentation of $\mathrm{Sp}_4(\mathbb{Z})$ as a quotient of B_6.

• For the proof we make use of the corresponding Steinberg group.

Steinberg groups - Generalities

• **R. Steinberg (1962):** For any irreducible root system Φ he defined the now-called Steinberg group by generators and relations. This group is by definition an extension of the simple complex algebraic group G of type Φ.

• **M. Stein (1971)** extended Steinberg's construction over any commutative ring R, leading to the Steinberg group $\mathrm{St}(\Phi, R)$.

• For $R = \mathbb{Z}$ (the ring of integers) $\mathrm{St}(\Phi, \mathbb{Z})$ has the following presentation:

- **Generators:** x_γ ($\gamma \in \Phi$).

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- **Generators:** x_γ $(\gamma \in \Phi)$.
- **Relations:** if $\gamma, \delta \in \Phi$ such that $\gamma + \delta \neq 0$, then

$$[x_\gamma, x_\delta] = \prod x_{i\gamma + j\delta}^{c_{i,j}^{\gamma,\delta}},$$

  where i and j are positive integers such that $i\gamma + j\delta$ belongs to Φ and the exponents $c_{i,j}^{\gamma,\delta}$ are integers depending only on the structure of the Chevalley group $G(\mathbb{Z})$.

The Steinberg group comes with a natural projection $\pi : \mathrm{St}(\Phi, \mathbb{Z}) \to G(\mathbb{Z})$.

Steinberg groups - The root system A_n

Let us start with the simplest case. Though simple, it is important because of its link with algebraic K-theory.

- The corresponding Chevalley group is $G(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Z})$.

- $\mathrm{SL}_n(\mathbb{Z})$ is generated by the elementary matrices

$$e_{i,j} = I_n + E_{i,j} \qquad (1 \leq i \neq j \leq n)$$

where $E_{i,j}$ is the $n \times n$ matrix which has all entries equal to 0 except the (i,j)-entry which is equal to 1.

- **The Steinberg group** $\mathrm{St}(A_n, \mathbb{Z})$. Often denoted by $\mathrm{St}_n(\mathbb{Z})$, it has a presentation with generators

$$x_{i,j} \qquad (1 \leq i \neq j \leq n)$$

and relations

$$[x_{i,j}, x_{k,\ell}] = \begin{cases} x_{i,\ell} & \text{if } j = k \text{ and } i \neq \ell, \\ 1 & \text{if } j \neq k \text{ and } i \neq \ell. \end{cases}$$

These relations are the "natural" ones between the generators $e_{i,j}$ of $\mathrm{SL}_n(\mathbb{Z})$.

Steinberg groups - The root system C_n

Th C_n-case is more involved because of its more complicated root system.

- Consider the Euclidean real vector space $\mathbb{R}^n$ and an orthonormal basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$.

The roots of the root system C_n are $\pm\varepsilon_i \pm \varepsilon_j$ (short roots) and $\pm 2\varepsilon_i$ (long roots), where $1 \leq i \neq j \leq n$.

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• **Generators of** $\mathrm{Sp}_{2n}(\mathbb{Z})$**:**

- $X_{i,j} = I_{2n} + E_{i,j} - E_{j+n,i+n}$ $(1 \leq i \neq j \leq n)$, corresponding to the root $\varepsilon_i - \varepsilon_j$,

Here $E_{i,j}$ is the $2n \times 2n$ matrix which has all entries equal to 0 except the (i,j)-entry which is equal to 1.

(These generators are obtained as follows: consider the Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{C})$ with its root space decomposition; in each root space take a generator and exponentiate it.)

• In order to obtain a presentation of the Steinberg group $\mathrm{St}(C_n, \mathbb{Z})$, we compute the commutators of all pairs of above generators.

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- $Y'_{i,j} = Y^T_{i,j}$ $(1 \leq i \neq j \leq n)$, corresponding to $-\varepsilon_i - \varepsilon_j$,
- $Z_i = I_{2n} + E_{i,i+n}$ $(1 \leq i \leq n)$, corresponding to $2\varepsilon_i$,
- $Z'_i = Z_i^T$ $(1 \leq i \leq n)$, corresponding to $-2\varepsilon_i$.

Here $E_{i,j}$ is the $2n \times 2n$ matrix which has all entries equal to 0 except the (i,j)-entry which is equal to 1.

(These generators are obtained as follows: consider the Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{C})$ with its root space decomposition; in each root space take a generator and exponentiate it.)

• In order to obtain a presentation of the Steinberg group $\mathrm{St}(C_n, \mathbb{Z})$, we compute the commutators of all pairs of above generators.

The symplectic Steinberg group

- **Presentation of the Steinberg group $\mathrm{St}(C_n, \mathbb{Z})$:**
  - **Generators:** $x_{i,j}$, $y_{i,j}$, $y'_{i,j}$, z_i, z'_i $(1 \leq i \neq j \leq n)$.

- The projection $\pi : \mathrm{St}(C_n, \mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z})$ is given by

$$\pi(x_{i,j}) = X_{i,j}\,, \quad \pi(y_{i,j}) = Y_{i,j}\,, \quad \pi(y'_{i,j}) = Y'_{i,j}\,, \quad \pi(z_i) = Z_i\,, \quad \pi(z'_i) = Z'_i\,.$$

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  - **Relations:** (the subscripts $i, j, k \in \{1, \ldots, n\}$ are pairwise distinct)

$$y_{i,j} = y_{j,i}\,, \qquad y'_{i,j} = y'_{j,i}\,,$$

$$[x_{i,j}, x_{j,k}] = x_{i,k}\,, \qquad [x_{i,j}, y_{j,k}] = y_{i,k}\,, \quad [x_{i,j}, y'_{i,k}] = y'^{-1}_{j,k},$$

$$[x_{i,j}, y_{i,j}] = z_i^2\,, \qquad [x_{i,j}, y'_{i,j}] = z'^{-2}_j,$$

$$[x_{i,j}, z_j] = z_i y_{i,j} = y_{i,j} z_i\,, \qquad [x_{i,j}, z'_i] = z'_j y'^{-1}_{i,j} = y'^{-1}_{i,j} z'_j\,,$$

$$[y_{i,j}, z'_i] = x_{j,i} z_j^{-1} = z_j^{-1} x_{j,i}\,, \qquad [y'_{i,j}, z_i] = x_{i,j}^{-1} z'^{-1}_j = z'^{-1}_j x_{i,j}^{-1}.$$

  All remaining pairs of generators commute, except $(x_{i,j}, x_{j,i})$, $(y_{i,j}, y'_{i,j})$ and (z_i, z'_i) for which we do not prescribe any relation.

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- **Matsumoto (1969):** Let $w_i = z_i z'^{-1}_i z_i \in \mathrm{St}(C_n, \mathbb{Z})$. The kernel of $\pi : \mathrm{St}(C_n, \mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z})$ is an infinite cyclic group generated by $w_i^4 = w_1^4$ for all i.

Lifting $\bar{f} : B_{2n+2} \to \mathrm{Sp}_{2n}(\mathbb{Z})$ to $\mathrm{St}(C_n, \mathbb{Z})$

- **Proposition (joint with François Digne).**

(a) There exists a unique homomorphism $\widetilde{f} : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$ *such that*

$$\widetilde{f}(\sigma_{2i}) = z_i'^{-1} \qquad (i = 1, \ldots, n),$$

$$\widetilde{f}(\sigma_{2i+1}) = z_i z_{i+1} y_{i,i+1}^{-1} \qquad (i = 1, \ldots, n-1),$$

$$\widetilde{f}(\sigma_1) = z_1 \,, \qquad \widetilde{f}(\sigma_{2n+1}) = z_n \,,$$

(b) The homomorphism $\widetilde{f}$ *lifts* $\bar{f} : B_{2n+2} \to \mathrm{Sp}_{2n}(\mathbb{Z})$*:*

$$\bar{f} = \pi \circ \widetilde{f}.$$

(c) The homomorphism $\widetilde{f}$ *is surjective if and only if* $n \leq 2$*.*

- We now return to the case $n = 2$.

The case $n = 2$ - Theorem 2 implies Theorem 1

- **Theorem 2.** *(a) The map $\widetilde{f} : B_6 \to \mathrm{St}(C_2, \mathbb{Z})$ is surjective.*

(b) Its kernel is the normal subgroup N of B_6 generated by the element

$$\omega = (\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5).$$

- Theorem 2 implies Theorem 1 (recalled below) in view of

$$\mathrm{Ker}\,(\pi : \mathrm{St}(C_2, \mathbb{Z}) \to \mathrm{Sp}_{2n}(\mathbb{Z})) = \langle w_1^4 \rangle \quad \text{and} \quad \widetilde{f}(\sigma_1\sigma_2\sigma_1) = w_1\,,$$

where $w_1 = z_1 z_1'^{-1} z_1$.

Theorem 1. *(a) The map $\bar{f} : B_6 \to \mathrm{Sp}_4(\mathbb{Z})$ is surjective.*

(b) Its kernel is the normal subgroup of B_6 generated by ω and $(\sigma_1\sigma_2\sigma_1)^4$.

The case $n = 2$ - Proof of Theorem 2

- **Recall.** Let N be the normal subgroup of B_6 generated by

$$\omega = (\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5).$$

- Theorem 2 is a consequence of the following two lemmas.

- **Lemma 1.** *We have* $\widetilde{f}(N) = 1$.

Proof. It suffices to check $\widetilde{f}(\omega) = 1$. Indeed,

$$\widetilde{f}(\omega) = w_1^2\, y_{1,2}\, w_1^{-2} y_{1,2} = y_{1,2}^{-1}\, y_{1,2} = 1.$$

- By Lemma 1 the map $\widetilde{f}$ induces a homomorphism $\widetilde{f} : B_6/N \to \mathrm{St}(C_2, \mathbb{Z})$. It is surjective.

Lemma 2. *There exists a homomorphism* $\varphi : \mathrm{St}(C_2, \mathbb{Z}) \to B_6/N$ *such that* $\varphi \circ \widetilde{f} = \mathrm{id}$.

Hence, $\widetilde{f} : B_6/N \to \mathrm{St}(C_2, \mathbb{Z})$ is also injective.

The case $n = 2$ - About Lemma 2

- **Recall.** Let N be the normal subgroup of B_6 generated by

$$\omega = (\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5).$$

- **Lemma 2.** *There exists a homomorphism* $\varphi : \mathrm{St}(C_2, \mathbb{Z}) \to B_6/N$ *such that* $\varphi \circ \widetilde{f} = \mathrm{id}$. *Modulo* N *we have*

$$\varphi(z_1) \equiv \sigma_1\,, \quad \varphi(z_2) \equiv \sigma_5\,, \quad \varphi(z_1') \equiv \sigma_2^{-1}, \quad \varphi(z_2') \equiv \sigma_4^{-1},$$

$$\begin{aligned}
\varphi(y_{1,2}) &\equiv \sigma_1\sigma_3^{-1}\sigma_5\,, \\
\varphi(y_{1,2}') &\equiv (\sigma_1\sigma_2\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)^{-1}(\sigma_1\sigma_2\sigma_5\sigma_4)^{-1}, \\
\varphi(x_{1,2}) &\equiv (\sigma_5\sigma_4)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_5\sigma_4)^{-1}, \\
\varphi(x_{2,1}) &\equiv (\sigma_1\sigma_2)(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2)^{-1}.
\end{aligned}$$

Proof. One checks that the image under φ of each defining relation of $\mathrm{St}(C_2, \mathbb{Z})$ is satisfied in B_6/N. For instance, for the relation $[y_{1,2}, z_1'] = x_{2,1}\, z_2^{-1}$, one has

$$\left[\varphi(y_{1,2}), \varphi(z_1')\right]^{-1} \varphi(x_{2,1})\, \varphi(z_2^{-1}) = \omega \in N.$$

- Note the ubiquity of the braid word $\sigma_1\sigma_3^{-1}\sigma_5$. We will generalize it in the next slide.

An element of the kernel of $\widetilde{f} : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$

Back to the general case $n \geq 3$. The following is joint work with François Digne.

• Let $\Delta_k^2 \in B_k$ be the positive braid generating the center of the group B_k of braids with k strands. It is defined inductively by $\Delta_2^2 = \sigma_1^2$ and

$$\Delta_{k+1}^2 = \Delta_k^2(\sigma_k\sigma_{k-1}\cdots\sigma_2\sigma_1)(\sigma_1\sigma_2\cdots\sigma_{k-1}\sigma_k).$$

Consider the following elements of the braid group B_{2n+2}:

$$\alpha_n = \Delta_3^2\Delta_5^2\cdots\Delta_{2n-1}^2$$

and

$$\beta_n = \sigma_1\sigma_3^{-1}\cdots\sigma_{2n+1}^{(-1)^n}.$$

• **Proposition (joint with F. Digne).** *For all $n \geq 2$ we have*

$$\widetilde{f}\left(\alpha_n\beta_n\alpha_n^{-1}\beta_n\right) = 1 \in \mathrm{St}(C_n, \mathbb{Z}).$$

Questions

I finish by mentioning work in progress with François Digne.

- We have just remarked that $\alpha_n\beta_n\alpha_n^{-1}\beta_n$ belongs to the kernel of

$$\widetilde{f} : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z}).$$

Observe that for $n = 2$,

$$\alpha_2\beta_2\alpha_2^{-1}\beta_2 = \omega = (\sigma_1\sigma_2\sigma_1)^2(\sigma_1\sigma_3^{-1}\sigma_5)(\sigma_1\sigma_2\sigma_1)^{-2}(\sigma_1\sigma_3^{-1}\sigma_5) \in B_6$$

generates the kernel of $\widetilde{f} : B_6 \to \mathrm{St}(C_2, \mathbb{Z})$ as a normal subgroup.

- **Question 1.** When $n \geq 3$, do $\alpha_n\beta_n\alpha_n^{-1}\beta_n$ and its conjugates generate the kernel of

$$\widetilde{f} : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})?$$

- **Question 2.** What is the image of $\widetilde{f} : B_{2n+2} \to \mathrm{St}(C_n, \mathbb{Z})$?

Merci pour votre attention

Thank you for your attention

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