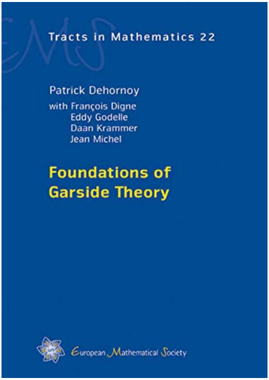
Trying to understand the intersection of parabolic subgroups after Cumplido, Gebhardt, Gonzalez and Wiest

Jean Michel (joint work with François Digne)

University Paris Diderot

Caen, 10th September 2021



The setting

We look for a setting which holds for a spherical Artin monoid, and for the dual monoid of a finite complex reflection group.

Let M be a Garside monoid with Garside element Δ, and with an additive length function $l : M \to \mathbb{N}$. We assume M is generated by the set S of its elements of length 1, which form a finite set (the atoms).

We call *standard parabolic* a submonoid stable by left and right divisors and left and right lcms.

We call $I \subset S$ *saturated* if it is the set of atoms of a standard parabolic submonoid (equivalently if $M_I := \langle I \rangle$ is standard parabolic).

In general for $I \subset S$ we denote by $\overline{I}$ the smallest saturated subset of atoms which contains I.

We assume that for any $I \subset S$ the right and left lcms of I are equal and equal to the lcm of $\overline{I}$.

We denote $\Delta_I = \Delta_{\overline{I}}$ that lcm. It is a Garside element for $M_{\overline{I}}$.

I-head and tail

For $b \in M$ we write $b = H(b)T(b)$ where $H(b)$ is the first term of the Garside normal form of b and $T(b)$ is the rest.

Proposition

For I saturated, any $b \in M$ has a unique maximal left-divisor $H_I(b)$ in M_I.

Proof.

The right lcm of two divisors of b in M_I is in M_I. □

We define $T_I(b)$ by $b = H_I(b)T_I(b)$. We say that b is I-reduced if $H_I(b) = 1$.

The ribbon category

We call *ribbon category* the category with objects the saturated subsets of S and maps $I \xrightarrow{g}$ when $g \in M$ is I-reduced and $I^g \subset S$. The target of this map is I^g. We will see that I^g is automatically saturated.

The ribbon category makes sense in a more general context, for example in any Artin group with the only condition that the monoids M_I be spherical.

Definition

For I saturated and $s \in S - I$, we define $v_{s,I} \in M$ and $J \subset I \cup \{s\}$ by the equalities $\Delta_I v_{s,I} = v_{s,I}\Delta_J = \Delta_{I\cup\{s\}}(= \Delta_{\overline{I\cup\{s\}}})$.

Note that since $\Delta_{\overline{I\cup\{s\}}}$ does an automorphism of $\overline{I \cup \{s\}}$, it conjugates a saturated part to another one.

For $a, b \in M$ we write $a \preccurlyeq b$ if a left-divides b.

Lemma (Atoms of the ribbon category)

Assume that $I \xrightarrow{g}$ is in the ribbon category and that $s \in S$ left-divides g. Then $v_{s,I} \preccurlyeq g$.

Proof.

We have $\Delta_I g = g\Delta_I^g$. Thus $s \preccurlyeq \Delta_I g$, thus $\Delta_{I\cup\{s\}} \preccurlyeq \Delta_I g$, thus $v_{s,I} = \Delta_I^{-1}\Delta_{I\cup\{s\}} \preccurlyeq g$. □

The fact that a map in the ribbon category is a product of such atoms implies that its target is saturated.

Lemma (I-head and I-tail preserved by ribbons)

Let $I \xrightarrow{g}$ be in the ribbon category and let $h \in M$. Then $T_I(gh) = gT_{I^g}(h)$ and $H_I(gh)^g = H_{I^g}(h)$.

Proof.

Let $s \in I$ and set $s' := s^g$. Both formulae clearly follow if we show that it is equivalent that $s \preccurlyeq gh$ or that $s' \preccurlyeq h$.
Now from $sg = gs'$ it follows that the right lcm of s and g divides sg, thus it must be equal to sg (because its length is at least $l(g)+1$). Thus $s \preccurlyeq gh$ is equivalent to $sg \preccurlyeq gh$ or equivalently $gs' \preccurlyeq gh$ which is finally equivalent to $s' \preccurlyeq h$. □

Proposition

If $I \xrightarrow{g}$ is in the ribbon category, so are all the terms of its Garside normal form.

Proof.

It is sufficient to prove that $I \xrightarrow{H(g)}$ and $I^{H(g)} \xrightarrow{T(g)}$ are in the ribbon category.
For the first fact, since $H_I(g) = 1$ implies $H_I(H(g)) = 1$ it is sufficient to prove that $I^{H(g)} \subset S$. Let $s \in I$. Then $s^g \in S$ is equivalent to $g \preccurlyeq sg$ which gives
$H(g) \preccurlyeq H(sg) = H(sH(g)) \preccurlyeq sH(g)$ which implies $s^{H(g)} \in S$.
For the second fact, since $(I^{H(g)})^{T(g)} = I^g \subset S$, it is sufficient to show that $T(g)$ is $I^{H(g)}$-reduced. By Lemma (I-head preserved) with $H(g), T(g)$ for g, h, we get that
$H_{I^{H(g)}}(T(g)) = H_I(H(g)T(g))^{H(g)} = 1$. □

Proposition

If $I \xrightarrow{g}$ and $I \xrightarrow{g'}$ are in the ribbon category, then so is $I \xrightarrow{\text{left-gcd}(g,g')}$

Proof.

It is clear that $H_I(\text{left-gcd}(g,g')) = 1$ so we have to show that $I^{\text{left-gcd}(g,g')} \subset S$. The proof is by induction on $\max(l(g), l(g'))$. If $\text{left-gcd}(g,g') \neq 1$, then there is some $a \in S$ such that $a \preccurlyeq g$ and $a \preccurlyeq g'$. By Lemma(atoms of the ribbon category) we have $v(a,I) \preccurlyeq g$ and $v(a,I) \preccurlyeq g'$. We conclude by induction by replacing g by $v(a,I)^{-1}g$ and g' by $v(a,I)^{-1}g'$. □

Proposition

If $I \xrightarrow{g}$ and $I \xrightarrow{g'}$ are in the ribbon category, then so is $I \xrightarrow{\text{right-lcm}(g,g')}$.

Proof.

We first show that $I^{\text{right-lcm}(g,g')} \subset S$. Indeed, if $g \preccurlyeq sg$, and $g' \preccurlyeq sg'$ then $\text{right-lcm}(g,g') \preccurlyeq s\,\text{right-lcm}(g,g')$. It remains to show that $k = \text{right-lcm}(g,g')$ is I-reduced. Note that Lemma (I-tail preserved) implies that if $I \xrightarrow{g}$ is in the ribbon category and $g \preccurlyeq k$ then $g \preccurlyeq T_I(k)$. It follows that g, g' left-divide $T_I(k)$, thus $k = \text{right-lcm}(g,g')$ divides $T_I(k)$ which proves that k is I-reduced. □

Proposition (Ribbon prefix)

For $I \subset S$, let $g \in M$ be I-reduced. Then there is a unique maximal prefix h of g in the ribbon category (with same source as g, that is such that $I^h \subset S$). If we denote $c_I(g)$ this prefix, $c_I(g)^{-1}g$ is $I^{c_I(g)}$-reduced and there is equivalence between:

(i) $c_I(g) = 1$.

(ii) *The left descent set of $\Delta_I g$ is I.*

Proof.

The existence of $c_I(g)$ is a consequence of the fact that the ribbon category is stable by right-lcms. The fact that $c_I(g)^{-1}g$ is $I^{c_I(g)}$-reduced is an immediate consequence of Lemma (preservation of head).

We finally prove the equivalence of (i) and (ii) by observing that $c_I(g) \neq 1$ is equivalent to the existence of $t \notin I$ such that $v_{t,I} \preccurlyeq g$, which is in turn equivalent to $\Delta_{I\cup\{t\}} \preccurlyeq \Delta_I g$ which is equivalent to t being in the left descent set of $\Delta_I g$. □

Proposition (Positive conjugacy)

Let $g \in M$, $b \in M_I$ be such that g is I-reduced and such that $b^g \in M$. Assume further that $\Delta_I \preccurlyeq b^i$ for some integer i. Then $I^g \subset S$.

Proof.

We first note that the assumption $b^g \in M$, that we write $bg = gu$ for some $u \in M$, implies $b^k g = gu^k$ for any $k > 0$; thus replacing b by some power we may assume $\Delta_I \preccurlyeq b$, say $b = \Delta_I v$.

We have thus the equality $\Delta_I vg = gu$. If $s \in S$, $s \preccurlyeq g$ it follows that $\Delta_{I\cup\{s\}}$ left-divides both sides thus $v_{s,I} \preccurlyeq vg$.

We now show by induction on the length of v that $v_{s,I} \preccurlyeq g$. We may write $v = tv'$ with t in I. Since vg is left-divisible by $v_{s,I}$ and by t, it is divisible by their right-lcm, which is equal to $tv_{s,I}$. Thus $v_{s,I} \preccurlyeq v'g$, and we conclude by induction.

We now conclude by induction on the length of g, since replacing g by $v_{s,I}^{-1}g$ and I by $J = I^{v_{s,I}}$ all the assumptions remain. □

We now denote by G the group of fractions of the Garside monoid M. We call standard parabolic subgroups the subgroups G_I generated by a standard parabolic monoid M_I, and parabolic subgroups the conjugates of the standard parabolic subgroups.

Definition

We say that $p^{-1}q$ is a left reduced fraction for $b \in G$ if $b = p^{-1}q$ with $p, q \in M$ and the left-gcd of p and q is trivial.

Symmetrically there are right reduced fractions pq^{-1}. "reduced fraction" will mean left reduced fraction.
The left reduced fraction for an element is unique; more precisely if $p^{-1}q$ is reduced and $p^{-1}q = p'^{-1}q'$ there exists d such that $p' = dp$ and $q' = dq$.
If an element of G_I has $p^{-1}q$ as its reduced fraction in G, then $p, q \in M_I$ since there is a reduced fraction $a^{-1}b$ in G_I and $a \succcurlyeq p$ and $b \succcurlyeq q$.

Lemma (Head in M_I)

Let $b \in G_I, g \in M$ be such that $b^g \in M$. Then $b^{H_I(g)} \in M_I$.

Proof.

Write $b^g = b'$, and write $g = H_I(g)T_I(g)$. Let $b^{H_I(g)} = pq^{-1}$ where the right-hand side is a right reduced fraction in G_I ($p, q \in M_I$). From the equality of the two fractions $T_I(g)b'T_I(g)^{-1} = pq^{-1}$ we deduce $q \preccurlyeq T_I(g)$ which implies $q = 1$ since $T_I(g)$ is I-reduced. □

We can now weaken the assumptions in Proposition (positive conjugacy)

Proposition (positive conjugate implies tail ribbon)

Let $b \in M_I, g \in M$ such that $b^g \in M$. Then $b^{H_I(g)} \in M_I$. Assume that for some $i > 0$ we have $\Delta_I^i \preccurlyeq b$. Then $I^{T_I(g)} \subset S$.

Proof.

By Lemma (head in M_I) $b^{H_I(g)}$ is still in M_I. Since $T_I(g)$ is I-reduced we can now apply Proposition (positive conjugacy) with b replaced by $b^{H_I(g)}$ and g replaced by $T_I(g)$ and we get the result. □

Lemma (conjugating Δ_I conjugates I)

Let Δ_I^k be central in M_I. If $(\Delta_I^k)^g \in M_J$ for $g \in M$, then $I^{T_I(g)} \subset J$.

Proof.

We have $(\Delta_I^k)^{H_I(g)} = \Delta_I^k$, thus $(\Delta_I^k)^{T_I(g)} \in M_J$. By the Proposition on positive conjugates we get $I^{T_I(g)} \subset S$, hence $I^{T_I(g)} \subset J$. □

Proposition (conjugate parabolics)

For two saturated subsets $I, J \subset S$ we have equivalence between:

(i) *There is $g \in G$ such that $G_I^g = G_J$.*

(ii) *There exist an integer $k > 1$ and $g \in G$ such that $(\Delta_I^k)^g = \Delta_J^k$.*

(iii) *There is $g \in G$ such that $I^g = J$*

Proof.

Clearly that (iii) implies (ii) for any k. Let us show that (ii) implies (iii). We can raise to some power the equality $(\Delta_I^k)^g = \Delta_J^k$ to ensure Δ_I^k is central in M_I. By multiplying g by some central power of Δ we may assume $g \in M$. By Lemma (conjugating Δ_I conjugates I) we have $I^{T_I(g)} = J' \subset J$, hence $\Delta_J^k = (\Delta_I^k)^g = (\Delta_I^k)^{T_I(g)} = \Delta_{J'}^k$. This implies $J' = J$ whence (iii). Since clearly (iii) implies (i), it remains to show that (i) implies (iii). Assume thus (i) and let $u = (\Delta_I^k)^g \in G_J$. By multiplying g^{-1} by some central power of Δ we may find $h \in M$ such that $u^h = \Delta_I^k$. Then by the Lemma (head in M_I) we have $u^{H_J(h)} \in M_J$. We thus have $(\Delta_I^k)^{T_J(h)^{-1}} \in M_J$. Multiplying by a central power of Δ we get $m \in M$ such that $(\Delta_I^k)^m \in M_J$. By the above Lemma we have $I^{T_I(m)} \subset J$. But h^{-1}, thus $T_J(h)^{-1}$, thus m, thus $T_I(m)$ conjugates G_I onto G_J hence $I^{T_I(m)} = J$. □

Corollary

It is equivalent that $g \in G$ conjugates G_I onto G_J, or that it conjugates some central power Δ_I^k to another Δ_J^k.

Proof.

We remark that in the proof of either (i) $\Rightarrow$ (iii) or (ii) $\Rightarrow$ (iii) of Proposition (conjugate parabolics) the element obtained from g is in the coset $G_I g G_J$ up to a central power of Δ, whence the result. □

Definition

If $P = G_I^g$ is a parabolic subgroup, we denote by z_P the element $(\Delta_I^k)^g$ where Δ_I^k is the smallest central power.

z_P depends only on P and not on I and g by the above Corollary.

Proposition

Let $P = G_I^b$ be a parabolic subgroup of G, where $I \subset S$ is saturated and $b \in M$. Define b' by $H_I(b)c_I(T_I(b))b' = b$ and J by $J = I^{c_I(T_I(b))}$. Then $b'^{-1}\Delta_J^k b'$ is the reduced fraction of z_P, where Δ_J^k is the smallest central power of Δ_J in M_J.

Proof.

We first remark that by definition we have $(\Delta_I^k)^b = z_P$. We may clearly replace b in this equality by $T_I(b)$. Let $c = c_I(T_I(b))$; we have $(\Delta_I^k)^c = \Delta_J^k$. We thus get $z_P = b'^{-1}\Delta_J^k b'$. We claim this is a reduced fraction. Indeed by construction $c_J(b') = 1$ thus by Proposition (ribbon prefix)(ii) the left descent set of $\Delta_J b'$ is J, thus the same is true for $\Delta_J^k b'$. Since b' is J-reduced, the fraction is reduced. □

Note that b' above is minimal such that ${}^{b'}P$ is standard, that is any $u \in M$ such that ${}^{u}P$ is standard is a left multiple of b', and G_J is a "canonical" standard parabolic subgroup conjugate to P.

Support

We call *support* of $b \in M$ the smallest saturated I such that $b \in M_I$.

For now on unless stated otherwise we assume M is a spherical Artin monoid, because this is the setting where we can prove the next proposition (where we replace the assumption Δ_I divides some power of b with $\mathrm{supp}(b) = I$). If we assume this proposition the rest works.

Proposition (Positive conjugate implies ribbon)

Let $g, b \in M$, $I = \mathrm{supp}(b)$ such that g is I-reduced and such that $b^g \in M$. Then $I \xrightarrow{g}$ is in the ribbon category, that is $I^g = \mathrm{supp}(b^g) \subset S$.

Proof.

We first show that we can reduce to the case where g is simple, by arguing by induction on the number of terms of the Garside normal form of g.

Proof continued

The assumption that $b^g \in M$ can be written $g \preccurlyeq bg$ from which it follows that $H(g) \preccurlyeq H(bg) = H(bH(g))$, in particular $H(g) \preccurlyeq bH(g)$, that is $b^{H(g)} \in M$; and for g to be $\operatorname{supp}(b)$-reduced, we certainly need that $H(g)$ is $\operatorname{supp}(b)$-reduced. If we know the theorem in the case where g is simple, it follows that $J := \operatorname{supp}(b)^{H(g)} \subset S$; and thus $b^{H(g)}$ has support J. We also have that $T(g)$ is J-reduced by Lemma (I-tail preserved by ribbons). Since $(b^{H(g)})^{T(g)} \in M$, we conclude by induction on the number of terms of the normal form of g that $J^{T(g)} \subset S$, whence the result since $J^{T(g)} = \operatorname{supp}(b)^g$.

We now show the theorem when g is simple. We write the condition that $b^g \in M$ as $bg = gu$ with $u \in M$.

We proceed by an induction on the length of b. For $s \in S$ dividing b, we write $b = sb'$. We have $s \preccurlyeq bg = gu$.

Proof continued

Lemma

Let $g \in M$ be simple and let $s \in S$, $u \in M$ be such that $s \not\preccurlyeq g$ but $s \preccurlyeq gu$. Then we have $u = u'tu''$ where $t \in S$ such that $sgu' = gu't$ is simple.

Proof.

First note that $s \preccurlyeq H(gu)$. Let $gu_1 = H(gu)$. Since in the Coxeter group s is in the left descent set of gu_1 but not in that of g we have by the exchange lemma $gu_1 = gu'tu_2$ where $t \in S$ and $sgu' = gu't$. Lifting back to M we get the lemma with $u'' = u_2T(gu)$. □

Proof continued.

Since s does not divide g, the lemma gives $u = u'tu''$ with $t \in S$ and $sgu' = gu't$. From $bg = sb'g = gu = gu'tu'' = sgu'u''$ we deduce $b'g = gu'u''$. Thus b', g satisfy the assumptions of the theorem, thus by induction on the length of b we have $\mathrm{supp}(b')^g \subset S$. We still have to prove that $s^g \in S$. This is already proven unless $s \notin \mathrm{supp}(b')$, which we assume now.
Since $\mathrm{supp}(b')^g \subset S$, we can write $b'g = gb''$ for some $b'' \in M$. Since $sgb'' = gu = gu'tu'' = sgu'u''$, we have $b'' = u'u''$. Now $g\,\mathrm{supp}(b'')g^{-1} \subset \mathrm{supp}(b')$, hence $v := gu'g^{-1} \in M_{\mathrm{supp}(b')}$. From $sgu' = gu't$ we get $svg = vgt$ which we write as $(v^{-1}sv)g = gt$. Since g is $\mathrm{supp}(b)$-reduced $l(v^{-1}svg) = l(v^{-1}sv) + l(g)$, whence $l(v^{-1}sv) = 1$. But $\mathrm{supp}(v) \subset \mathrm{supp}(b')$ so that $s \not\subset (\mathrm{supp}\, v)$, hence $l(v^{-1}sv) = 1$ implies $v^{-1}sv = s$, thus $s^g = t$. □

Proposition (Support remains full)

Let $b \in M$ such that $\mathrm{supp}(b) = S$ and let $g \in G$ such that $u := b^g \in M$. Then $\mathrm{supp}(u) = S$.

Proof.

We first observe that we can assume $g^{-1} \in M$, since multiplying g by some central power of Δ_S does not change the assumptions. Assume that $\mathrm{supp}(u) = J \subsetneq S$. We will derive a contradiction. Write $g^{-1} = ag'$ where $g' \in M$ is J-reduced and $a = H_J(g^{-1})$. From Lemma (head in M_I) we get that $u^a \in M_J$ and $(u^a)^{g'} = b$. Since g' is J-reduced, we may apply Proposition (positive conjugate implies ribbon) and we deduce that $\mathrm{supp}(b) \subset J^g \subset S$, a contradiction. □

Proposition

Let $J, K \subset S$. Then there is equivalence between

(i) *There exists $b \in M_J$ of support J and $g \in G$ such that $b^g \in M_K$.*

(ii) *There exists $g \in M$ such that $J^g \subset K$.*

Proof.

It is clear that (ii) implies (i), so we have to prove that (i) implies (ii). We may assume that $g \in M$ by multiplying it by a suitable central power of Δ. We may use lemma (head in M_I) which tells us that replacing g by g' and b by $b^{H_J(g)}$ where $g = H_J(g)g'$ we may assume that $b \in M_J$ and that g is J-reduced; note that we still have $\mathrm{supp}(b) = J$ by Proposition (support remains full). We then apply Proposition (positive conjugate implies ribbon) to deduce that $J^g \subset S \cap M_K = K$. □

Proposition

Let $b \in M$; then $G_{\mathrm{supp}(b)}$ is the unique minimal parabolic subgroup of G which contains b.

Proof.

Suppose $b \in G_J^g$, another parabolic, which is minimal containing b; thus there exists $u \in G_J$ such that $u^g = b$. Multiplying by a central power of Δ, we may assume $g \in M$. Applying Lemma (head in M_I) we may assume that $u \in M_J$ and that g is J-reduced. We may also assume that $\mathrm{supp}(u) = J$ otherwise G_J^g is not minimal (since $G_{\mathrm{supp}(u)}^g \ni b$ already). By Proposition (positive conjugate implies ribbon) we have that $J^g = K \subset S$ for some K, and $K = \mathrm{supp}(u^g) = \mathrm{supp}(b)$. Thus $G_J^g = G_{\mathrm{supp}(b)}$. □

It follows that if for $b, g \in G$ we have $b^g \in M$, then the unique minimal parabolic subgroup containing b is ${}^g G_{\mathrm{supp}(b^g)}$.

Reversing

For the next frames M is an arbitrary Garside monoid with finitely many simples.

Definition

If $p^{-1}q$ is the reduced fraction for $b \in G$ we define

- the support of b by $\mathrm{supp}(b) = \mathrm{supp}(p) \cup \mathrm{supp}(q)$,
- the denominator of b by $\mathrm{den}(b) = p$,
- the reverse of b by $\mathrm{rev}(b) = {}^{\mathrm{den}(b)}b = qp^{-1}$.

Note that the support agrees with the previous definition when $b \in M$.
Note also that, the Garside element is central and p and q are simple and both non trivial, then rev is cycling.

We denote by $l_\Delta(b)$ the number of factors in the Garside normal form of $b \in M$.

Proposition (reversing circuits)

(i) *If $b = p^{-1}q$ and $\mathrm{rev}(b) = p'^{-1}q'$ are reduced fractions, we have $l_\Delta(p') \leq l_\Delta(p)$ and $l_\Delta(q') \leq l_\Delta(q)$.*

(ii) rev *is ultimately periodic, that is, for $b \in G$, there exists $j > i \geq 0$ such that $\mathrm{rev}^i(b) = \mathrm{rev}^j(b)$.*

We write $\mathrm{RC}(b)$ for the "reversing circuit" of $b \in G$, that is the set of elements which can be reached from b by iterating rev and belong to a period of rev, and we set $\mathrm{RC}(G) = \cup_{b\in G} \mathrm{RC}(b)$.

For $b \in \mathrm{RC}(G)$ and $i \geq 1$ let
$\mathrm{den}^{(i)}(b) = \mathrm{den}(\mathrm{rev}^{i-1}(b)) \ldots \mathrm{den}(\mathrm{rev}(b))\,\mathrm{den}(b)$, so that
${}^{\mathrm{den}^{(i)}(b)}b = \mathrm{rev}^i(b)$.
If $g \in M$ is such that $b^g \in \mathrm{RC}(G)$; for $i \geq 1$ define
$\mathrm{rev}^{(i)}(g) = \mathrm{den}^{(i)}(b) g (\mathrm{den}^{(i)}(b^g))^{-1}$.

Proposition (reversing circuits...)

(iii) $\mathrm{rev}^{(i)}$ *is ultimately periodic, more precisely, there exists* $j > i > 0$ *such that* $b = \mathrm{rev}^i(b) = \mathrm{rev}^j(b)$, $b^g = \mathrm{rev}^i(b^g) = \mathrm{rev}^j(b^g)$, $\mathrm{rev}^{(i)}(g) = \mathrm{rev}^{(j)}(g)$.

(iv) *We have* $l_\Delta(\mathrm{den}^{(i)}(b)) \leq i l_\Delta(\mathrm{den}(b))$.

(v) *If* b *has a conjugate in* M *then* $\mathrm{RC}(b) \subset M$. *If* i *is minimal such that* $\mathrm{rev}^i(b) \in M$, *then* $i \leq l(\Delta)$ *and* $\mathrm{den}^{(i)}(b)$ *is the shortest element conjugating* b *into* M.

Proof.
Since the left-gcd of p' and q' is trivial, $l_\Delta(p')$ is the least m such that $\Delta^m p'^{-1} q' \in M$. But $qp^{-1} = p'^{-1}q'$, thus $l_\Delta(p') \leq l_\Delta(p)$. Similarly, from $pq^{-1} = q'^{-1}p'$ we get $l_\Delta(q') \leq l_\Delta(q)$, whence (i). Moreover, since the number of divisors of a fixed power of Δ is finite we get the ultimate periodicity of rev, whence (ii).
Let us prove (iii). We first prove that $\mathrm{rev}^{(1)}(g)$ is in M. We have $\mathrm{rev}^{(1)}(g) = \mathrm{den}(b) g\,\mathrm{den}(b^g)^{-1}$ and we have to prove that $\mathrm{den}(b^g)$ right-divides $\mathrm{den}(b)g$. Let $b = p^{-1}q$ and $b^g = u^{-1}v$ be reduced fractions. We have thus $(pg)^{-1}(qg) = u^{-1}v$ so that $pg \succcurlyeq u$ (and $qg \succcurlyeq v$). Since $\mathrm{den}(b^g) = u$ and $\mathrm{den}(b) = p$, we have $\mathrm{rev}^{(1)}(g) \in M$.
If $g, h \in M$ are such that b^g and b^{gh} are in $\mathrm{RC}(G)$, then we have $\mathrm{rev}^{(1)}(g)\,\mathrm{rev}^{(1)}(h) = \mathrm{rev}^{(1)}(gh)$, hence $\mathrm{rev}^{(1)}$ preserves left divisibility. Since $\mathrm{rev}^{(1)}(\Delta) = \Delta$ we deduce that if $g \preccurlyeq \Delta^m$ for some m, then $\mathrm{rev}^{(i)}(g) \preccurlyeq \Delta^m$ for all i.

Proof continued

Since b and b^g are in $\mathrm{RC}(G)$, there exists i_0 such that $\mathrm{rev}^{i_0}(b) = b$ and $\mathrm{rev}^{i_0}(b^g) = b^g$. Since the number of divisors in M of Δ^m is finite, there exist $i < j$, two multiples of i_0, such that $\mathrm{rev}^{(i)}(g) = \mathrm{rev}^{(j)}(g)$, whence (iii).

(iv) comes from the fact that $l_\Delta(\mathrm{den}(\mathrm{rev}^j(b))) \le l_\Delta(\mathrm{den}(b))$ for any $j \ge 0$ by (i).

We prove (v). First note that if $p^{-1}q$ is the reduced fraction for b and $g \in M$ is such that ${}^g b = c \in M$, then $g \succcurlyeq p$ since $p^{-1}q = g^{-1}(cg)$ and $p^{-1}q$ is a reduced fraction. Take g such that $l(g)$ is minimal such that $gbg^{-1} \in M$. Let us write $g = g'p$. Now we have $gp^{-1}qg^{-1} = g'qp^{-1}g'^{-1} = g' \,\mathrm{rev}(b) g'^{-1}$ and g' has minimal length such that $g' \,\mathrm{rev}(b) g'^{-1} \in M$, since if $l(g'') < l(g')$ and $g'' \,\mathrm{rev}(b) g''^{-1} = g''qp^{-1}g''^{-1} \in M$ then $l(g''p) < l(g'p) = l(g)$ and $g''qp^{-1}g''^{-1} = g''pb(g''p)^{-1} \in M$, a contradiction with the minimality of g.

Proof continued.

We can iterate the process, replacing b with $\mathrm{rev}(b)$ and g with g'. Let $g^{(1)} := g'$ and $g^{(i)}$ be the element obtained at the i-th step. Since $g^{(i)} \prec g^{(i-1)}$ we will get eventually $g^{(i)} = 1$ (for the first i such that $\mathrm{rev}^i(b) \in M$) which shows that $RC(b) \subset M$ and also that $g = \mathrm{den}^{(i)}(b)$.

We now claim that $H(g') \prec H(g)$. Assume by contradiction $H(g') = H(g)$. Write $g = H(g)T(g)$. From $gp^{-1}q = cg$ with $c \in M$ we get $g'q = cg$, hence $H(g) = H(g') \preccurlyeq cg$, whence $H(g) \preccurlyeq cH(g)$. Writing $cH(g) = H(g)c_1$ and cancelling $H(g)$, we obtain $T(g)p^{-1}qT(g)^{-1} = c_1 \in M$, which contradicts the minimality of g since $l(T(g)) < l(g)$.

We have thus proved that $i \le l(H(g))$. □

Note that in (iii) we can assume $j - i > 1$ by adding to j the length of a period if necessary.

Now M is again a spherical Artin monoid.

Theorem

Let $b \in \mathrm{RC}(G), g \in M$ such that $b^g \in \mathrm{RC}(G)$. Let $I = \mathrm{supp}(b)$ and $J = \mathrm{supp}(b^g)$. Then $I^{T_I(g)} = J$.

Proof.

We first note that if $b \in M$, then $b^g \in M$ also and we get the result by Proposition (positive conjugate implies ribbon).
If $b^{-1} \in M$, since by definition we have $\mathrm{den}(b) = b^{-1}$, we get $\mathrm{rev}(b) = \mathrm{rev}(b^{-1})^{-1}$. Since b and b^{-1} have same support, the property for b^{-1} gives the result for b.

□

Proof continued.

In general, by Proposition (reversing circuits)(iii), it is possible to choose $j > i > 0$ such that $\mathrm{rev}^i(b) = \mathrm{rev}^j(b) = b$, $\mathrm{rev}^i(b^g) = \mathrm{rev}^j(b^g) = b^g$ and

$$(\mathrm{den}^{(i)}(b))^{-1} g \,\mathrm{den}^{(i)}(b^g) = (\mathrm{den}^{(j)}(b))^{-1} g \,\mathrm{den}^{(j)}(b^g). \qquad (17)$$

Since $\mathrm{rev}^i(b) = b$, we have $\mathrm{den}^{(j)}(b) = \mathrm{den}^{(j-i)}(b)\,\mathrm{den}^{(i)}(b)$ and similarly $\mathrm{den}^{(j)}(b^g) = \mathrm{den}^{(j-i)}(b^g)\,\mathrm{den}^{(i)}(b^g)$. Thus the equation gives $\mathrm{den}^{(j-i)}(b)^g = \mathrm{den}^{(j-i)}(b^g)$. Let $u_k(b) = \mathrm{den}^{(k)}(b)b$. We still have $u_{j-i}(b)^g = u_{j-i}(b^g)$. Now note that if $p^{-1}q$ is the reduced fraction for b and $p'^{-1}q'$ that for $\mathrm{rev}(b)$ we have $u_2(b) = \mathrm{den}(\mathrm{rev}(b))\,\mathrm{den}(b)b = p'pb = p'q = \text{left-lcm}(p, q)$, the last equality since $p'q = q'p$; thus for $k \geq 2$ the element $u_k(b)$ is in M and has same support as b and similarly $u_k(b^g) \in M$ with same support as b^g. Up to increasing j we may assume $k = j - i \geq 2$ and apply Proposition (positive conjugate implies tail ribbon) to $u_k(b)^g = u_k(b^g)$ and we get the result. □

Proposition (Minimal parabolic)

Let $b \in \mathrm{RC}(G)$. Then $G_{\mathrm{supp}(b)}$ is the unique minimal parabolic subgroup of G which contains b.

Proof.

Suppose $b \in G_J^g$, another parabolic, which is minimal containing b; let $u = {}^g b \in G_J$. By multiplying on the left g by some element of G_J we can replace u by some element in $\mathrm{RC}(G) \cap G_J$.
We may also assume that $g \in M$ up to multiplying g by some even power of Δ. We may then apply proposition (positive conjugate implies tail ribbon) to conclude that $\mathrm{supp}(u)^{T_{\mathrm{supp}(u)}(g)} = \mathrm{supp}(b)$.
We also have $\mathrm{supp}(u) = J$ otherwise G_J^g is not minimal (since $G^g_{\mathrm{supp}(u)} \ni b$ already), thus $G_J^g = G_J^{T_J(g)} = G_{\mathrm{supp}(b)}$. □

Corollary

Any element of G is contained in a unique minimal parabolic subgroup $P_{\min}(b)$.

Proof.

Let $b \in G$ and $b^g \in \mathrm{RC}(G)$; then $P_{\min}(b) := {}^g G_{\mathrm{supp}(b^g)}$ is the unique minimal parabolic subgroup containing b. □

Definition

For $b \in G$ we define $\varphi(b) = l(\Delta_I)$ if $G_I^g = P_{\min}(b)$.

Note that $\varphi(b)$ is a well-defined invariant of the conjugacy class of b by Proposition (conjugate parabolics). Thanks to Proposition (minimal parabolic), another way of defining $\varphi(b)$ is as $l(\Delta_{\mathrm{supp}(b')})$ if b' is any element of $\mathrm{RC}(b)$.

Proposition

Let $\beta \in G$ and $I \subset S$ be such that for any m, we have $\varphi(\beta\Delta_I^m) \leq l(\Delta_I)$. Then $\beta \in G_I$.

Proof.

We first show that for m large enough, we have $\mathrm{RC}(\beta\Delta_I^m) \subset M$. Let $\tilde{l}$ be the topological length on G (for a fraction $p^{-1}q$, it is equal to $l(q) - l(p)$), which is invariant by conjugacy. If β_m is any element of $\mathrm{RC}(\beta\Delta_I^m)$, we thus have $\tilde{l}(\beta_m) = \tilde{l}(\beta) + mn$, where $n = l(\Delta_I) = \tilde{l}(\Delta_I)$. □

Proof continued

Now let us define $l_\Delta(b)$ for an arbitrary $b \in G$ as $l_\Delta(p) + l_\Delta(q)$ where $p^{-1}q$ is a reduced fraction for b. Then we have $l_\Delta(\beta\Delta_I^m) \leq l_\Delta(\beta) + m$. The same majoration holds for $l_\Delta(\beta_m)$ by Proposition (reversing circuits)(i).

Now if $\beta_m \notin M$ and has reduced fraction say $p_m^{-1}q_m$, no term of the normal form of p_m or q_m is equal to $\Delta_{\mathrm{supp}(\beta_m)}$ so any term x of these normal forms satisfies $l(x) < l(\Delta_{\mathrm{supp}(\beta_m)}) \leq n$, the last inequality by the definition of n and Proposition (Minimal parabolic). It follows that $l(p_m) + l(q_m) \leq (n-1)l_\Delta(\beta_m) \leq (n-1)(l_\Delta(\beta) + m)$. Now we have $\tilde{l}(\beta) + mn = \tilde{l}(\beta_m) \leq l(p_m) + l(q_m) \leq (l_\Delta(\beta) + m)(n-1)$, a contradiction when m is large enough, which refutes the assumption $\beta_m \notin M$.

For m large enough, since $\beta\Delta_I^m$ has a conjugate in M, by Proposition (reversing circuits)(v) we have $\mathrm{rev}^{l(\Delta)}(\beta\Delta_I^m) \in M$. Let us denote by c_m the element $\mathrm{den}^{(l(\Delta))}(\beta\Delta_I^m)$ of Proposition (reversing circuits)(iv); this element conjugates $\beta\Delta_I^m$ to $\mathrm{rev}^{l(\Delta)}(\beta\Delta_I^m)$.

Proof continued

If $a^{-1}b$ is the reduced fraction for β, since $\mathrm{den}(\beta\Delta_I^m) \preccurlyeq a$ we get by Proposition (reversing circuits)(iv) that $l_\Delta(c_m)$ is bounded by $l(\Delta)l_\Delta(a)$ which is independent of m. Thus we have $(\beta\Delta_I^m)^{c_m} \in M$ where $c_m \in M$ has l_Δ bounded by some number h independent of m.

In the following lemma we write $x_1, \ldots, x_r$ for the r terms of the normal form of the element $x_1 x_2 \cdots x_r$.

Lemma

Let $c_1, \ldots, c_r, \underbrace{\Delta_J, \ldots, \Delta_J}_{l \text{ terms}}, d_1, \ldots, d_t$ be the normal form of an element of $x \in M$, with $J \subset S$, and let $c \in M$. Then the normal form of cx has at least $l - l_\Delta(c)$ terms equal to Δ_J.

Proof.

By induction on $l_\Delta(c)$, it suffices to prove that when c is simple, the normal form of cx has a similar shape to the form of x given in the statement, with $l-1$ terms equal to Δ_J in the middle. The normal form of $cc_1 \ldots c_r\Delta_J$ is computed recursively from left to right by replacing c, c_1 by $H(cc_1), T(cc_1)$, then applying this process to $T(cc_1)$ and c_2, etc..., eventually applying the process to y and Δ_J giving $H(y\Delta_J), T(y\Delta_J)$; thus the normal form is of the form $c'_1, \ldots, c'_{r'}, j$ where the right descent set of $c'_{r'}$ contains J and $j \in M_J$. It follows that $cx = c'_1 \ldots c'_{r'}\Delta_J^{l-2} j'\Delta_J d_1 \ldots d_t$ where $c'_1, \ldots, c'_{r'}, \underbrace{\Delta_J, \ldots, \Delta_J}_{l-2 \text{ terms}}$ is a normal form and j' is the conjugate of j by Δ_J^{l-2}. We now claim that the normal form of $j'\Delta_J d_1 \ldots d_t$ is of the form $\Delta_J, d'_1, \ldots, d'_{t'}$. Indeed $H(j'\Delta_J d_1 \ldots d_t) = H(j'\Delta_J) = \Delta_J$. Thus the normal form of cx is $c'_1, \ldots, c'_{r'}\underbrace{\Delta_J, \ldots, \Delta_J}_{l-1 \text{ terms}}, d'_1, \ldots, d'_{t'}$ which proves that the induction works. □

Proof continued.

y Proposition (Ribbon prefix) if we write $c_m = H_I(c_m)e_m d_m$ where $e_m = c_I(T_I(c_m))$ then $J_m := I^{e_m} \subset S$ and the left descent set of $\Delta_{J_m} d_m$ is J_m. It follows that the normal form of $\Delta_{J_m}^m d_m$ has its $m-1$ first terms equal to Δ_{J_m}. If $a^{-1}b$ is the reduced fraction for β, we apply the Lemma to $b\Delta_I^m c_m = bH_I(c_m)^{\Delta_I^m} e_m \Delta_{J_m}^m d_m$ with $J = J_m$, $c = bH_I(c_m)^{\Delta_I^m} e_m$ and $x = \Delta_{J_m}^m d_m$ and, using that $l_\Delta(c)$ is bounded by $h + l_\Delta(b)$, conclude that in the normal form of $b\Delta_I^m c_m$ there are at least $m-1-h-l_\Delta(b)$ terms equal to Δ_J. Since $c_m^{-1}a^{-1}b\Delta_I^m c_m \in M$, the element ac_m must divide the first $l_\Delta(ac_m)$ terms of the normal form of $b\Delta_I^m c_m$. Thus the last $l_\Delta(b\Delta_I^m c_m) - l_\Delta(ac_m)$ terms of the normal form of $b\Delta_I^m c_m$ are a right divisor of $(\beta\Delta_I^m)^{c_m} \in M$, and since $l_\Delta(ac_m)$ is bounded by $h + l_\Delta(a)$, for m large enough these terms contain at least one copy of Δ_{J_m}. Thus the support of $(\beta\Delta_I^m)^{c_m}$ contains J_m. Since $l(\Delta_{J_m}) = l(\Delta_I) = n$, the support cannot be greater than J_m, by the assumption on $\varphi(\beta\Delta_I^m)$, so is equal to J_m. It follows in particular that $d_m \in G_{J_m}$, so that $e_m^{-1}\beta\Delta_I^m e_m \in G_{J_m}$, thus $\beta\Delta_I^m \in G_{{}^{e_m}J_m} = G_I$, thus finally $\beta \in G_I$. □

Theorem

The intersection of two parabolic subgroups of G is a parabolic subgroup.

Proof.

Let P and Q be two parabolic subgroups, and let $H \in P \cap Q$ be an element with $\varphi(H)$ maximal amongst elements of $P \cap Q$. We claim that $P \cap Q = P_{\min}(H)$. Up to conjugating H, P, Q we may (and we will) assume that $P_{\min}(H)$ is the standard parabolic G_I. Let $\beta \in P \cap Q$; since $\beta\Delta_I^m \in P \cap Q$ we have $\varphi(\beta\Delta_I^m) \le \varphi(H) = l(\Delta_I)$ thus by the above Proposition we have $\beta \in G_I$. □