ORDERING BRAIDS: IN MEMORY OF PATRICK DEHORNOY

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With the untimely passing of Patrick Dehornoy in September 2019, the world of mathematics lost a brilliant scholar who made profound contributions to set theory, algebra, topology, and even computer science and cryptography. And I lost a dear friend and a strong influence in the direction of my own research in mathematics. In this article I will concentrate on his remarkable discovery that the braid groups are left-orderable, and its consequences, and its strong influence on my own research. I'll begin by describing how I learned of his work.

1. When I met Patrick

In the late 1990's I had been working on a conjecture of Joan Birman [1] regarding the braid groups B_n and the (larger) monoids SB_n of singular braids. I remind the reader that B_n has the presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$ and $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ when $|i-j| = 1$.

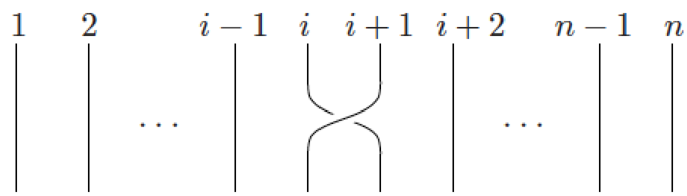

Figure 1: The Artin generator σ_i

In addition to the usual Artin generators σ_i in which the i-th strand passes below the next strand, SB_n has elements τ_i in which those strands intersect each other. In the study of SB_n and influenced by Vassiliev theory, Birman proposed the following mapping:

$$\sigma_i \longrightarrow \sigma_i$$
$$\tau_i \longrightarrow \sigma_i - \sigma_i^{-1}$$

For this to make sense, the target needs to be not just B_n, but rather the group ring $\mathbb{Z}B_n$. She conjectured that this map $SB_n \to \mathbb{Z}B_n$ is injective.

Investigating this problem with a student at the time, Jun Zhu, we were making calculations in $\mathbb{Z}B_n$, including cancellations such as $ab = ac \implies b = c$. Such an implication is not valid in a ring if there are zero divisors involved. It's possible that $a(b-c) = 0$ but neither a nor $b-c$ is zero. So we always had to check that we were not cancelling zero-divisors. This led me to the question of whether zero-divisors actually exist in $\mathbb{Z}B_n$. I asked various experts, such as Vaughan Jones and Joan Birman, whether they knew the answer to that question. Joan replied that some work of Dehornoy might be relevant. I found the paper [5] in which Patrick

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proved the result that there is a strict total ordering of the braid group B_n which is invariant under left-multiplication. To my delight, this answered my question, as I also learned that left-orderable groups have the property that their integral group rings have no zero divisors. (This is conjectured to be true more generally for torsion-free groups.)

The main point of the paper [5], as stated there, was to prove:

Theorem 1. *There is an effective algorithm for deciding whether a given identity is or is not a consequence of the left distributivity identity* $x(yz) = (xy)(xz)$.

This was motivated by the study of elementary embeddings in set theory. I confess that I did not (and still do not) completely understand that paper. In hopes of learning more, I contacted Patrick and arranged to meet him to discuss his ordering, during a planned trip across the Atlantic to attend a conference in England. I think it was summer of 1998. We met over dinner in Paris, accompanied by his charming wife, Arlette, who is also a mathematician. Patrick very patiently tried to explain his ordering of B_n. Although I did grasp some of the details, I did not understand the crucial fact upon which the ordering depends, namely Theorem 3 stated below. His proof involved a partial action of B_n on the Cartesian product of n copies of a self-distributive set, which in turn can be ordered. These were beyond my comprehension, and I think quite mysterious to topologists.

2. Patrick's influence on me

A few days later, at the conference in Sussex – Roger Fenn was one of the organizers – I gave a talk and explained what I knew about Dehornoy's ordering of B_n, why I found it exciting, and asked if we could fashion an argument that was understandable to topologists. A number of us got thinking about this problem, sometimes over pints, and we eventually, weeks later, managed to find an alternative proof [12], using the interpretation of B_n as the mapping class group of the n-punctured disk. If a braid in B_n is interpreted as a homeomorphism of the round disk in the complex plane with diameter the real interval $[0, n+1]$, fixed on the boundary and permuting the punctures $\{1, 2, \dots, n\}$, one looks at the image of the real axis. This is only defined up to homeomorphism fixed on the punctures and the boundary, but one can insist the image of the real line intersects the real line minimally. The result is called a curve diagram. The positive cone is then all braids for which the curve diagram (reading left to right) first departs from the real line into the *upper* half-plane. Remarkably, this coincides with the set of σ-positive braids as will be defined later (Definition 2).

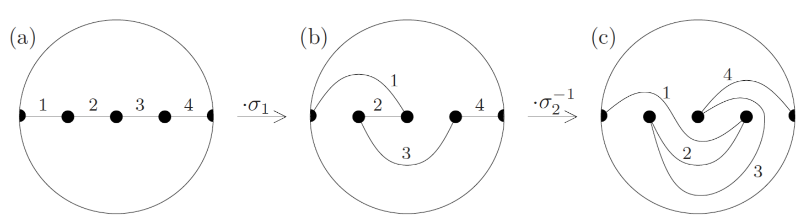

Figure 2: Curve diagrams for (a) the identity, (b) σ_1 and (c) $\sigma_1\sigma_2^{-1}$.

Incidentally, Zhu and I did not solve Birman's conjecture except for the cases $n \leq 3$, but it was solved a few years later in the affirmative by Luis Paris [20].

This experience, and Patrick's influence, changed the thrust of my own research. I had not even heard of orderable groups before learning of his braid ordering. The seed had been planted, and I've spent many happy years since then studying ordered groups and their connections with topology. This includes two books, [8] and [9], which were mostly written by Patrick, but with contributions by Ivan Dynnikov, Bert Wiest and myself. I spent many pleasant days in Patrick's home office in Evreux working with him, occasionally being on the phone with Bert or emailing Ivan, putting together those books. Our friendship grew accordingly.

3. Patrick's many talents

As we worked together, I came to appreciate Patrick's many talents, besides his obvious talents as a mathematician. He had impressive musical skills at the piano. He was an expert builder: much of his home was built by himself. He had a very caring personality and a great interest in philosophy. We had many discussions on that subject.

One of his hidden talents came to the fore in Caen in 2007. Patrick had nominated me for an honorary degree from the University of Caen, and to my shock (and delight) the nomination was successful. The recipients (as well as the nominators) are expected to wear academic gowns and make a short speech at the very formal occasion of the awards. My French is terrible, but I worked hard to prepare a speech *en français*, to the effect that one of my mathematical heros, Henri Poincaré, had been a professor in Caen, which made the award extra special to me. Then, to the great surprise of myself and others, Patrick gave a speech *in Latin!* He had learned Latin many years ago and was still fluent enough to give the speech without written notes, in keeping with the medieval tone of the gathering.

4. Patrick and set theory

I greatly admired Patrick's understanding of the issues of modern set theory. In 2014 he visited Vancouver as Distinguished Scholar, in a program co-sponsored by the French Consulate, the Pacific Institute for the Mathematical Sciences, and the Peter Wall Institute for Advanced Studies. Besides seminar talks and formalities, the culmination of the visit was a special public lecture. Here is the abstract of Patrick's fabulous talk "Set Theory: the last 50 years."

"At the interface of Mathematics, Computer Science, and Philosophy, Set Theory is both a fascinating subject and the victim of several misunderstandings: after the great successes in the first half of the XXth century, Set Theory was (mistakenly) thought to be a universal dogma resulting in well-known educational damages and to have come to an end, with a few mysterious questions due to remain open forever. The aim of the lecture will be to present a more accurate view of what Set Theory is, namely a theory of infinity and what it is not. Starting from a historical approach and putting the emphasis on Cantor's celebrated Continuum Problem, we shall explain what is the meaning of the remarkable results established by Goedel and by Cohen. But, then, and mainly, we shall present a few results of modern Set Theory as it developed after Cohen, a most ignored topic in spite of wonderful achievements. In particular, we shall explain how some new axioms by and by acquired a status of mathematical truths, inviting everyone to develop his own reflection about truth and infinity in mathematics."

It was indeed a most memorable talk. One of the main themes was large cardinals, which cannot be proven to exist in standard axioms of set theory. Indeed, Patrick's first argument for the existence of the σ-ordering assumed the existence of certain large cardinals. Again in Patrick's words [6], recalling the left-distributive axiom $x(yz) = (xy)(xz)$,

"The linear ordering of the braid group has been derived from a linear ordering on the free left-distributive system with one generator. The first complete proof of the existencee of the latter ordering occurred in a paper by R. Laver about large cardinals [16], and a large cardinal axiom from set theory was explicitly used to prove the ordering's irreflexivity (or the absence of cycle for the left divisibility relation in the terminology of the present text). This gave rise of course to the question of whether the large cardinal axiom was needed for the property. Actually it is not, as we shall see .... The point here is that considerations of highly infinite objects (elementary embeddings) have led to an *intuition* that ended in results of the most constructive spirit (for example a new algorithm for braid word comparison)."

5. Properties of Dehornoy's ordering

To return to the braid ordering, Dehornoy gives the following definition:

Definition 2. *A braid $\beta \in B_n$ is σ_i-positive if it has an expression in which σ_i occurs with only positive exponents, and no σ_j appears for $j < i$. Call β σ-positive if it is σ_i-positive for some i with $1 \leq i \leq n-1$.*

Let $P \subset B_n$ denote the set of all σ-positive n-braids, Dehornoy proved the following highly nontrivial fact.

Theorem 3 ([5])**.** *For every braid $\beta \in B_n$, exactly one of the following holds: $\beta \in P, \beta^{-1} \in P, \beta = 1$.*

The following is easy to see:

Proposition 4. *The product of two n-braids in P is again in P, that is, P is a sub-semigroup of B_n.*

Then the σ-ordering $<$ is defined by

$$\alpha < \beta \iff \alpha^{-1}\beta \in P.$$

The terminology "σ-ordering" is due to Patrick, in modesty I think. Most other people call it the Dehornoy ordering. From the properties of the positive cone $P = \{\beta \in B_n \mid 1 < \beta\}$ mentioned above, it is easy to check:

Corollary 5. *The relation $<$ is a strict total ordering of B_n satisfying $\alpha < \beta$ iff $\gamma\alpha < \gamma\beta$.*

This ordering has many interesting properties.

Proposition 6. *The Dehornoy ordering is a discrete ordering of B_n, that is there is a smallest braid which is greater than the identity, namely σ_{n-1}.*

To see this, first observe that σ_{n-1} is σ-positive. Moreover, if there were an n-braid β such that $1 < \beta < \sigma_{n-1}$ we would conclude that $\beta^{-1}\sigma_{n-1}$ is σ-positive. By Theorem 3, β^{-1} is not σ-positive. This implies that $\beta^{-1}\sigma_{n-1}$ must be σ_{n-1}-positive, and therefore a positive power of σ_{n-1}. This leads to a contradiction.

It follows that there is also a greatest element of B_n less than the identity in the Dehornoy ordering, namely σ_{n-1}^{-1}. Thus we see that every braid has an immediate successor in the order, and also an immediate predecessor.

By contrast, certain subgroups of B_n, are *densely* ordered by the same ordering, that is any two distinct braids in the subgroup have another braid in the subgroup strictly between them. Notice that all the generators σ_i are conjugate in B_n, for example $(\sigma_1\sigma_2)\sigma_1(\sigma_1\sigma_2)^{-1} = \sigma_2$. Therefore the abelianization of B_n is infinite cyclic, and the commutator subgroup $[B_n, B_n]$ consists of all braids whose exponent sum, in terms of the σ_i, is zero.

Theorem 7 ([3])**.** *If $n \geq 3$ the commutator subgroup $[B_n, B_n]$ of B_n is densely ordered in the restriction of the Dehornoy ordering. In particular, $[B_n, B_n]$ contains no smallest σ-positive element.*

Some other subgroups of B_n also have this density property, for example it is shown in [3] that the kernel of the Burau representation, for dimensions in which it is known to be unfaithful ($n \geq 5$).

The subgroup PB_n of pure braids, which has index $n!$ in B_n, consists of braids whose strands begin and end at the same level. That is, PB_n is the kernel of the homomorphism from B_n to the symmetric group Σ_n sending σ_i to the permutation interchanging i and $i+1$. PB_n does have a smallest positive element, namely σ_{n-1}^2.

Theorem 8 ([15])**.** *PB_n can be given a strict total ordering (not the restriction of Dehornoy's ordering) which is invariant under multiplication on both sides.*

One way to see this is to note that by a process called Artin combing, PB_n can be expressed as a semidirect product of free groups, which in turn possess 2-sided orderings. The following can be easily checked, either by drawing a picture or using the braid relations.

Example 9. *Let $\beta = \sigma_1\sigma_2^{-1}$ and $\gamma = \sigma_1\sigma_2\sigma_1$ be braids in $B_n, n \geq 3$. Then $\gamma^{-1}\beta\gamma = \beta^{-1}$.*

Proposition 10. *For $n \geq 3, B_n$ cannot admit a strict total ordering which is invariant under multiplication on both sides.*

That's because $1 < \beta \iff \beta^{-1} < 1$ always holds for a left (or right) invariant order. A 2-sided ordering of B_n would be invariant under conjugation. An equation $\gamma^{-1}\beta\gamma = \beta^{-1}$ therefore cannot hold (for $\beta \neq 1$) in a bi-orderable group. In fact more is true. The following is proved in [22] and independently in [10].

Theorem 11. *For $n \geq 3$, no 2-sided ordering of PB_n can be extended to a left-invariant ordering of B_n.*

Let B_n^+ denote the *positive braid monoid*, that is, all n-braids which can be expressed as words in the σ_i using no negative exponents. It is clear that all braids in $B_n^+ \setminus \{1\}$ are greater than the identity in the Dehornoy ordering.

Theorem 12 ([17])**.** *The restriction of Dehornoy's ordering to B_n^+ is a well-ordering, meaning that every nonempty subset has a least element.*

The order type is known to be rather large. In his doctoral thesis [2], Burckel showed that the Dehornoy well-ordering of B_n^+ has order type $\omega^{\omega^{n-2}}$.

6. Applications to knot theory

The main connection between braid groups and knot theory is the closure of a braid $\hat{\beta}$, in which the ends of the strands of β are connected without any further crossings. Every knot or link can be realized as the closure of some braid, according to a theorem of Alexander.

Let $\Delta_n^2 = (\sigma_1\sigma_2\cdots\sigma_{n-1})^n \in B_n$ be the (square of the) Garside element. We may abbreviate $\Delta_n^2 = \Delta^2$ when the braid index n is understood from the context. Then Δ^2 is a pure braid, is a generator of the infinite-cyclic center of B_n, if $n > 2$, and its powers are cofinal in the Dehornoy ordering of B_n. It corresponds to a "full twist." Malyutin and Netsvetaev showed that braids sufficiently far from the identity in the ordering behave nicely under closure.

Theorem 13 ([19])**.** *Suppose $n \geq 3$ and $\beta \in B_n$ is not in the interval $[\Delta^{-4}, \Delta^4]$ with respect to the Dehornoy order. Then the link closure $\hat{\beta}$ is prime, non-split and nontrivial.*

A braid, considered as a mapping class of the disk with n punctures, has an associated Thurston classification, being periodic, reducible or pseudo-Anosov. For braids far enough from the identity in the Dehornoy order, these concepts correspond exactly the knot type, according to the following result of Tetsuya Ito.

Theorem 14 ([14])**.** *Suppose $\beta \in B_n$, that $\hat{\beta}$ is a knot and β is not in the interval $[\Delta^{-4}, \Delta^4]$ with respect to the Dehornoy order. Then $\hat{\beta}$ is a torus knot iff β is periodic, a satellite knot iff β is reducible, and a hyperbolic knot iff β is pseudo-Anosov.*

The genus $g(K)$ of the knot $K \subset S^3$ is the least genus of all oriented surfaces in S^3 bounded by the knot. The Dehornoy order gives a lower bound to knot genus:

Theorem 15 ([13])**.** *Suppose $n \geq 3$ and $\beta \in B_n$ satisfies $\Delta^{2m} < \beta$ or $\beta < \Delta^{-2m}$. If $K = \hat{\beta}$ is a knot, then $m < g(K) + 1$.*

As already mentioned, every knot or link arises as the closure of a braid, but an important and often difficult question is: what is the *least* n such that a given link is the closure of an n-strand braid? This minimum is called the *braid index* of the knot or link. Again, Dehornoy's ordering sheds light on the braid index problem. A recent result of Feller and Hubbard gives the following, which had been conjectured by Malyutin and Netsvetaev.

Theorem 16 ([11])**.** *Fix an integer $n \geq 2$, and suppose $\beta \in B_n$ satisfies $\Delta^{2n} < \beta$ or $\beta < \Delta^{-2n}$. Then the closure $\hat{\beta}$ of β does not occur as the closure of a braid with fewer than n strands.*

7. The space of orderings

There are many possible left-orderings of B_n, for $n \geq 3$. Although the Dehornoy ordering plays a special rôle, it is by no means the only one.

Theorem 17. *If $n \geq 3$, then B_n has uncountably many distinct left-invariant orderings.*

It is not difficult to check that B_n has infinitely many left-orders, so this follows from a theorem of Peter Linnell [18], that for any group G, the set of left-orderings of G is either finite or uncountably infinite. For any group G, the set $LO(G)$ of all left-orderings has a natural topology, as described in [23]. A typical neighbourhood of an ordering is to specify a finite set of inequalities that hold in the ordering (alternatively, choose a finite set of elements greater than the identity). The corresponding neighbourhood consists of all possible left-orderings in which those inequalities still hold. This makes $LO(G)$ into a topological space which is compact, Hausdorff, and totally disconnected [23].

$LO(G)$ may have isolated points. For countable groups G, if $LO(G)$ is infinite and has no isolated points, then it is homeomorphic with the Cantor set. G acts on $LO(G)$ by conjugation: given $\prec \in LO(G)$ and $h \in G$ one can define the conjugate ordering $\prec_h \in LO(G)$ by $f \prec_h g \iff h^{-1}fh \prec h^{-1}gh$ (which by left-invariance is equivalent to $fh \prec gh$). It is easy to see that the mapping $\prec \to \prec_h$ is a homeomorphism of $LO(G)$. More generally, any automorphism of G acts on $LO(G)$ in a similar manner.

Dubrovin and Dubrovina described orderings of $B_n, n \geq 3$ with the property that their positive cones are *finitely generated* as sub-semigroups. It follows that they represent isolated points in the space $LO(B_n)$. For the case $n = 3$ we can take the positive cone P_{DD} of the ordering $<_{DD}$ to be the set of all braids which are either σ_1-positive or else σ_2-negative. It's an interesting exercise to check that P_{DD} is generated by $\sigma_1\sigma_2$ and σ_2^{-1} as a monoid, and is therefore the only ordering of B_3 in which those elements are positive.

Theorem 18. *If $n \geq 3$, the space $LO(B_n)$ contains isolated points. However, the Dehornoy ordering is not isolated in $LO(B_n)$; in fact it is a limit point of its conjugates.*

See, for example [9], p. 269 for a proof of the latter part of the theorem. Another group of interest is B_∞, which can be thought of as the direct limit, or union of the B_n, under the natural inclusions $B_n \subset B_{n+1}$. B_∞ has a presentation with infinitely many generators σ_i, for $i \in \mathbb{N}$ and the usual braid relations.

Proposition 19. *$LO(B_\infty)$ has no isolated points, and therefore is homeomorphic with the Cantor set. Every left-ordering is a limit point of its conjugates.*

To see this, consider a finite number of positive elements $F \subset B_\infty$ for a left-ordering $\prec$. Then $F \subset B_N$ for some finite N. Then for $i > N$, σ_i commutes with all the braids in F, and therefore the braids in F are positive in the conjugate ordering $\prec_{\sigma_i}$. In other words, $\prec_{\sigma_i}$ is in the neighbourhood of $\prec$ defined by F. It remains to check that at least some of the $\prec_{\sigma_i}$ are distinct from $\prec$ as orderings of B_∞. If that were not the case, then on the subgroup $\langle \sigma_{N+1}, \sigma_{N+2}, \dots \rangle$ we could conclude that $\prec$ was both right and left invariant, which is not possible, as $\langle \sigma_{N+1}, \sigma_{N+2}, \dots \rangle \cong B_\infty$ is not bi-orderable.

8. Le Paradis des mathématiciens

One of Patrick's most unusual (and amusing) accomplishments is the video "Le Paradis des mathématiciens" [7]. He is the director and also plays the leading role of a mathematician who dies and goes to heaven, where he can look down at events on earth through a telescope. He waits several centuries fearing that his beloved mathematical work had been forgotten. Finally a student of the future discovers his book ([9] in fact), and finds a theorem of Patrick's is exactly what he needed to construct his "turbulence algebra." Patrick sees this through the telescope and there is joy again in paradise. The reader is strongly encouraged to view it: https://vimeo.com/205778279

This story, and the production of it, is a witness to one more of Patrick's strong points: his wonderful sense of humor. Even now, I sometimes picture him up there with his telescope.

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