Braids, Orderings and Minimal Volume Hyperbolic 3-manifolds

Dale Rolfsen

University of British Columbia

September 2021

Ordered groups

This talk is dedicated to the memory of Patrick Dehornoy. It represents joint work with Eiko Kin.

Ordered groups

This talk is dedicated to the memory of Patrick Dehornoy. It represents joint work with Eiko Kin.

A left-ordering of a group G is a strict total ordering $<$ of its elements so that

$$g < h \implies fg < fh$$

for all $f, g, h \in G$.

Ordered groups

This talk is dedicated to the memory of Patrick Dehornoy. It represents joint work with Eiko Kin.

A left-ordering of a group G is a strict total ordering $<$ of its elements so that

$$g < h \implies fg < fh$$

for all $f, g, h \in G$. If also $g < h \implies gf < hf$, the ordering is called a bi-ordering.

Ordered groups

If $<$ is a left-ordering of G, the positive cone $P = \{g \in G | 1 < g\}$ satisfies:
(1) if $p, q \in P$ then $pq \in P$, that is P is a sub-semigroup.
(2) For each $g \in G$ exactly one of $g \in P, g^{-1} \in P$ or $g = 1$ holds.

Ordered groups

If $<$ is a left-ordering of G, the positive cone $P = \{g \in G | 1 < g\}$ satisfies:
(1) if $p, q \in P$ then $pq \in P$, that is P is a sub-semigroup.
(2) For each $g \in G$ exactly one of $g \in P, g^{-1} \in P$ or $g = 1$ holds.
Moreover, given a subset P satisfying (1) and (2), we can define a left-ordering of G by declaring $g < h \iff h^{-1}g \in P$.

Ordered groups

If $<$ is a left-ordering of G, the positive cone $P = \{g \in G | 1 < g\}$ satisfies:
(1) if $p, q \in P$ then $pq \in P$, that is P is a sub-semigroup.
(2) For each $g \in G$ exactly one of $g \in P, g^{-1} \in P$ or $g = 1$ holds.
Moreover, given a subset P satisfying (1) and (2), we can define a left-ordering of G by declaring $g < h \iff h^{-1}g \in P$.
P defines a bi-ordering by the same formula iff it also satisfies
(3) if $p \in P$ and $g \in G$, then $gpg^{-1} \in P$.

Ordered groups

Left- and bi-orderable groups have special properties. They are torsion-free and satisfy the zero-divisor conjecture: If R is a ring with no zero divisors and G is a left-orderable group, then the group ring $R[G]$ has no zero-divisors.

Ordered groups

Left- and bi-orderable groups have special properties. They are torsion-free and satisfy the zero-divisor conjecture: If R is a ring with no zero divisors and G is a left-orderable group, then the group ring $R[G]$ has no zero-divisors.
Moreover, the only units of $R[G]$ are the "trivial" ones rg with r a unit of R and $g \in G$.

Ordered groups

Bi-orderable groups have unique roots, meaning that $f^n = g^n$ for some $n > 0$ implies $f = g$.

Ordered groups

Bi-orderable groups have unique roots, meaning that $f^n = g^n$ for some $n > 0$ implies $f = g$. They also have no generalized torsion elements, meaning a nonidentity element for which a nontrivial product of its conjugates is the identity.

Ordered groups

Bi-orderable groups have unique roots, meaning that $f^n = g^n$ for some $n > 0$ implies $f = g$. They also have no generalized torsion elements, meaning a nonidentity element for which a nontrivial product of its conjugates is the identity.
Left- and bi-orderability are local properties, that is, a group G has the property iff each of its finitely-generated subgroups has.

Ordered groups

Bi-orderable groups have unique roots, meaning that $f^n = g^n$ for some $n > 0$ implies $f = g$. They also have no generalized torsion elements, meaning a nonidentity element for which a nontrivial product of its conjugates is the identity.
Left- and bi-orderability are local properties, that is, a group G has the property iff each of its finitely-generated subgroups has.

Proposition

An abelian group is bi-orderable iff it's torsion-free.

Ordered groups

Bi-orderable groups have unique roots, meaning that $f^n = g^n$ for some $n > 0$ implies $f = g$. They also have no generalized torsion elements, meaning a nonidentity element for which a nontrivial product of its conjugates is the identity.
Left- and bi-orderability are local properties, that is, a group G has the property iff each of its finitely-generated subgroups has.

Proposition

An abelian group is bi-orderable iff it's torsion-free.

On the other hand, there are many torsion-free (nonabelian) groups which fail to be left-orderable.

Ordered groups

Theorem

Free groups are bi-orderable.

Ordered groups

Theorem

Free groups are bi-orderable.

To see this consider a free group F, which we may assume finitely-generated, and consider the lower central series $F = F_1 \supset F_2 \supset F_3 \supset \cdots$, defined inductively by $F_{i+1} = [F, F_i]$.

Ordered groups

Theorem

Free groups are bi-orderable.

To see this consider a free group F, which we may assume finitely-generated, and consider the lower central series $F = F_1 \supset F_2 \supset F_3 \supset \cdots$, defined inductively by $F_{i+1} = [F, F_i]$. As is well-known, $\cap_{n=1}^{\infty} F_i = \{1\}$. and the lower-central quotients F_i/F_{i+1} are finitely-generated free abelian groups, so we may choose a bi-ordering for each of those quotients.

Ordered groups

Theorem

Free groups are bi-orderable.

To see this consider a free group F, which we may assume finitely-generated, and consider the lower central series $F = F_1 \supset F_2 \supset F_3 \supset \cdots$, defined inductively by $F_{i+1} = [F, F_i]$. As is well-known, $\cap_{n=1}^{\infty} F_i = \{1\}$. and the lower-central quotients F_i/F_{i+1} are finitely-generated free abelian groups, so we may choose a bi-ordering for each of those quotients.
Call an element $1 \neq x \in F$ positive if its class $\{x\} \in F_i/F_{i+1}$ is positive in the chosen ordering of F_i/F_{i+1}, where i is the greatest integer such that $x \in F_i$.

Ordered groups

Theorem

Free groups are bi-orderable.

To see this consider a free group F, which we may assume finitely-generated, and consider the lower central series $F = F_1 \supset F_2 \supset F_3 \supset \cdots$, defined inductively by $F_{i+1} = [F, F_i]$. As is well-known, $\cap_{n=1}^{\infty} F_i = \{1\}$. and the lower-central quotients F_i/F_{i+1} are finitely-generated free abelian groups, so we may choose a bi-ordering for each of those quotients.
Call an element $1 \neq x \in F$ positive if its class $\{x\} \in F_i/F_{i+1}$ is positive in the chosen ordering of F_i/F_{i+1}, where i is the greatest integer such that $x \in F_i$. It's routine to check that this positive cone defines a bi-ordering of F. Such an ordering of F will be called a "standard" ordering of F. There are many others.

Ordered groups

Patrick Dehornoy famously defined a left-ordering on the braid groups B_n, which has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations: $\sigma_i \sigma_j = \sigma_j \sigma_i$ when $|i - j| > 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

Ordered groups

Patrick Dehornoy famously defined a left-ordering on the braid groups B_n, which has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.
For $n \geq 3$ the braid group B_n is NOT bi-orderable. Indeed the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ implies that $(\sigma_1\sigma_2)^3 = (\sigma_2\sigma_1)^3$ whereas $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$.

Ordered groups

Patrick Dehornoy famously defined a left-ordering on the braid groups B_n, which has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations: $\sigma_i \sigma_j = \sigma_j \sigma_i$ when $|i - j| > 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.
For $n \geq 3$ the braid group B_n is NOT bi-orderable. Indeed the braid relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ implies that $(\sigma_1 \sigma_2)^3 = (\sigma_2 \sigma_1)^3$ whereas $\sigma_1 \sigma_2 \neq \sigma_2 \sigma_1$. Thus B_n does not have unique roots.

Ordered groups

Patrick Dehornoy famously defined a left-ordering on the braid groups B_n, which has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations: $\sigma_i \sigma_j = \sigma_j \sigma_i$ when $|i - j| > 1$ and $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.
For $n \geq 3$ the braid group B_n is NOT bi-orderable. Indeed the braid relation $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ implies that $(\sigma_1 \sigma_2)^3 = (\sigma_2 \sigma_1)^3$ whereas $\sigma_1 \sigma_2 \neq \sigma_2 \sigma_1$. Thus B_n does not have unique roots.
On the other hand, the pure braid groups P_n are bi-orderable.

Ordered groups

Patrick Dehornoy famously defined a left-ordering on the braid groups B_n, which has generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ when $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.
For $n \geq 3$ the braid group B_n is NOT bi-orderable. Indeed the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ implies that $(\sigma_1\sigma_2)^3 = (\sigma_2\sigma_1)^3$ whereas $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. Thus B_n does not have unique roots.
On the other hand, the pure braid groups P_n are bi-orderable.
It is known that no bi-ordering of P_n extends to a left-ordering of B_n.

Braid groups

Artin observed that the braid group B_n acts on the free group F_n by automorphisms, and indeed this representation $B_n \to Aut(F_n)$ is faithful. It is convenient to think of braids acting on the right:

$$x \to x^\beta$$

where $x \in F_n, \beta \in B_n$. Then we have $x^{(\beta\gamma)} = (x^\beta)^\gamma$.

Braid groups

Artin observed that the braid group B_n acts on the free group F_n by automorphisms, and indeed this representation $B_n \to Aut(F_n)$ is faithful. It is convenient to think of braids acting on the right:

$$x \to x^\beta$$

where $x \in F_n, \beta \in B_n$. Then we have $x^{(\beta\gamma)} = (x^\beta)^\gamma$.
Specifically, if the generators of F_n are $x_1, \ldots, x_n$, the Artin action of the generator σ_i is:
$x_{i+1} \to x_i, \quad x_i \to x_i x_{i+1} x_i^{-1}$ and $x_j \to x_j$ if $j \notin \{i, i+1\}$.

Order-preserving braids

Definition

A braid $\beta \in B_n$ is order-preserving if there is a bi-ordering $<$ of F_n preserved by β, that is, $x < y \iff x^\beta < y^\beta$.

Order-preserving braids

Definition

A braid $\beta \in B_n$ is order-preserving if there is a bi-ordering $<$ of F_n preserved by β, that is, $x < y \iff x^\beta < y^\beta$.

Note that the identity braid is order-preserving, and the inverse of an order-preserving braid is also order-preserving.

Order-preserving braids

Definition

A braid $\beta \in B_n$ is order-preserving if there is a bi-ordering $<$ of F_n preserved by β, that is, $x < y \iff x^\beta < y^\beta$.

Note that the identity braid is order-preserving, and the inverse of an order-preserving braid is also order-preserving.

Proposition

The Artin generator σ_i is NOT order-preserving.

Order-preserving braids

Definition

A braid $\beta \in B_n$ is order-preserving if there is a bi-ordering $<$ of F_n preserved by β, that is, $x < y \iff x^\beta < y^\beta$.

Note that the identity braid is order-preserving, and the inverse of an order-preserving braid is also order-preserving.

Proposition

The Artin generator σ_i is NOT order-preserving.

Indeed, if there were a bi-ordering $<$ of F_n preserved by σ_i we may assume $x_i < x_{i+1}$.

Order-preserving braids

Definition

A braid $\beta \in B_n$ is order-preserving if there is a bi-ordering $<$ of F_n preserved by β, that is, $x < y \iff x^\beta < y^\beta$.

Note that the identity braid is order-preserving, and the inverse of an order-preserving braid is also order-preserving.

Proposition

The Artin generator σ_i is NOT order-preserving.

Indeed, if there were a bi-ordering $<$ of F_n preserved by σ_i we may assume $x_i < x_{i+1}$. Applying σ_i we would conclude that $x_i x_{i+1} x_i^{-1} < x_i$. Since $<$ is conjugation-invariant, we'd conclude the contradiction $x_{i+1} < x_i$.

Order-preserving braids

Theorem

Every pure braid is order-preserving.

Order-preserving braids

Theorem

Every pure braid is order-preserving.

To see this, note that the action of a pure braid on the rank n free group F, induces the identity on the abelianization $\mathbb{Z}^n = F/[F, F]$. One then proves that it also acts by the identity on all the lower-central quotients. Therefore, it preserves every standard ordering of F.

Order-preserving braids

Theorem

Every pure braid is order-preserving.

To see this, note that the action of a pure braid on the rank n free group F, induces the identity on the abelianization $\mathbb{Z}^n = F/[F,F]$. One then proves that it also acts by the identity on all the lower-central quotients. Therefore, it preserves every standard ordering of F.

Corollary

The square σ_i^2 of each Artin generator is order-preserving.

Corollary

For every braid $\beta \in B_n$ some power β^k is order-preserving.

Order-preserving braids

Theorem

Every pure braid is order-preserving.

To see this, note that the action of a pure braid on the rank n free group F, induces the identity on the abelianization $\mathbb{Z}^n = F/[F,F]$. One then proves that it also acts by the identity on all the lower-central quotients. Therefore, it preserves every standard ordering of F.

Corollary

The square σ_i^2 of each Artin generator is order-preserving.

Corollary

For every braid $\beta \in B_n$ some power β^k is order-preserving.

Order-preserving braids

If $\beta \in B_n$ define the "braided link" $\mathrm{br}(\beta) = \hat{\beta} \cup A$ to be the union of the braid closure and the braid axis in S^3.

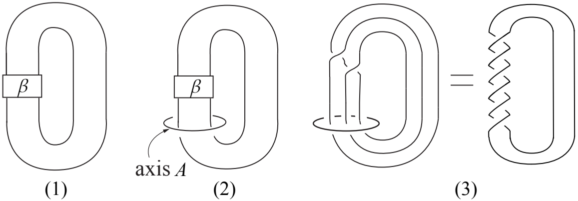


Figure: (1) Closure $\widehat{\beta}$. (2) $\mathrm{br}(\beta) = \widehat{\beta} \cup A$. (3) $\mathrm{br}(\sigma_1\sigma_2)$ is equivalent to the $(6,2)$-torus link.

Order-preserving braids

If $\beta \in B_n$ define the "braided link" $\mathrm{br}(\beta) = \hat{\beta} \cup A$ to be the union of the braid closure and the braid axis in S^3.

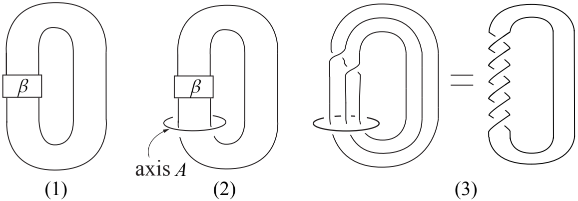


Figure: (1) Closure $\widehat{\beta}$. (2) $\mathrm{br}(\beta) = \widehat{\beta} \cup A$. (3) $\mathrm{br}(\sigma_1\sigma_2)$ is equivalent to the $(6, 2)$-torus link.

Order-preserving braids

Theorem

The braid $\beta \in B_n$ *is order-preserving if and only if the fundamental group of the complement* $S^3 \setminus \mathrm{br}(\beta)$ *is bi-orderable.*

Order-preserving braids

Theorem

The braid $\beta \in B_n$ is order-preserving if and only if the fundamental group of the complement $S^3 \setminus \mathrm{br}(\beta)$ is bi-orderable.

Corollary

$\beta \in B_n$ is OP iff each conjugate of β is OP iff $\beta\Delta_n^2$ is OP, where Δ_n^2 is the full twist.

Order-preserving braids

Theorem

The braid $\beta \in B_n$ is order-preserving if and only if the fundamental group of the complement $S^3 \setminus \mathrm{br}(\beta)$ is bi-orderable.

Corollary

$\beta \in B_n$ is OP iff each conjugate of β is OP iff $\beta\Delta_n^2$ is OP, where Δ_n^2 is the full twist.

Corollary

For any braid β, The group $\pi_1(S^3 \setminus \mathrm{br}(\beta))$ has a finite-index normal subgroup which is bi-orderable.

Order-preserving braids

Theorem

The braid $\beta \in B_n$ is order-preserving if and only if the fundamental group of the complement $S^3 \setminus \mathrm{br}(\beta)$ is bi-orderable.

Corollary

$\beta \in B_n$ is OP iff each conjugate of β is OP iff $\beta\Delta_n^2$ is OP, where Δ_n^2 is the full twist.

Corollary

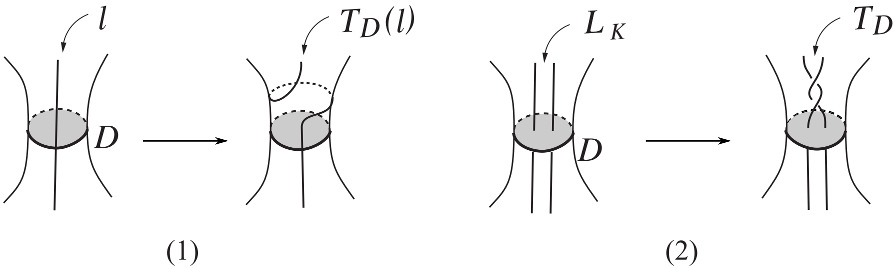
For any braid β, The group $\pi_1(S^3 \setminus \mathrm{br}(\beta))$ has a finite-index normal subgroup which is bi-orderable.

Corollary

Suppose $\beta \in B_n$ and $\gamma \in B_m$ satisfy $S^3 \setminus \mathrm{br}(\beta) \cong S^3 \setminus \mathrm{br}(\gamma)$, then β is OP iff γ is OP.

Order-preserving braids

If a link has an unknotted component, one can perform a disk twist along a disk bounded by that component. The result is a homeomorphism of the link complement, which might well change the link itself.



Order-preserving braids

Proposition

For each $k \geq 2$*, the braid* $\sigma_1\sigma_2\cdots\sigma_k\sigma_1$ *is order-preserving.*

Order-preserving braids

Proposition

For each $k \geq 2$, the braid $\sigma_1\sigma_2\cdots\sigma_k\sigma_1$ is order-preserving.

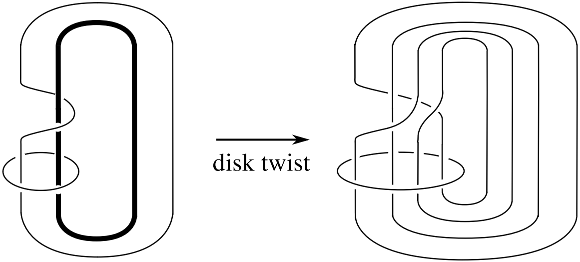


Figure: nth power of the disk twist converts the braided link of σ_1^2 to that of $\sigma_1\sigma_2\cdots\sigma_{n+1}\sigma_1$. ($n = 2$ in this case.)

Order-preserving braids

Let OP_n denote the set of order-preserving braids in B_n. For $n \geq 3$ we observe that OP_n is NOT closed under multiplication.

Order-preserving braids

Let OP_n denote the set of order-preserving braids in B_n. For $n \geq 3$ we observe that OP_n is NOT closed under multiplication.
Indeed we just saw that $\sigma_1\sigma_2\sigma_1 \in OP_n$, and $\sigma_1^{-2} \in OP_n$, being a pure braid. But their product is $\sigma_1\sigma_2\sigma_1^{-1}$, which is not order-preserving, being conjugate to σ_2.

Order-preserving braids

Let OP_n denote the set of order-preserving braids in B_n. For $n \geq 3$ we observe that OP_n is NOT closed under multiplication.
Indeed we just saw that $\sigma_1\sigma_2\sigma_1 \in OP_n$, and $\sigma_1^{-2} \in OP_n$, being a pure braid. But their product is $\sigma_1\sigma_2\sigma_1^{-1}$, which is not order-preserving, being conjugate to σ_2.
On the other hand OP_n contains the finite-index subgroup P_n of pure braids. We also have:

Proposition

The set OP_n generates B_n.

In other words, every braid is a product of order-preserving braids.

Order-preserving braids

There is a tensor product operation $B_m \times B_n \to B_{m+n}$ defined as follows

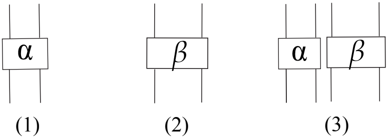


Figure: (1) $\alpha \in B_m$. (2) $\beta \in B_n$. (3) $\alpha \otimes \beta \in B_{m+n}$.

Theorem

$\alpha \otimes \beta$ is order-preserving if and only if both α and β are order-preserving.

Order-preserving braids

There is a tensor product operation $B_m \times B_n \to B_{m+n}$ defined as follows

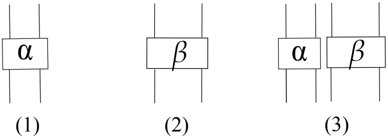


Figure: (1) $\alpha \in B_m$. (2) $\beta \in B_n$. (3) $\alpha \otimes \beta \in B_{m+n}$.

Theorem

$\alpha \otimes \beta$ is order-preserving if and only if both α and β are order-preserving.

Corollary

Suppose $n < k$ and $\beta \in B_n \subset B_k$ under the natural inclusion. Then β is OP in B_n iff it is OP in B_k.

Small hyperbolic 3-manifolds

We now turn to applications to the orderability of fundamental groups of certain minimal-volume orientable hyperbolic 3-manifolds.

Small hyperbolic 3-manifolds

We now turn to applications to the orderability of fundamental groups of certain minimal-volume orientable hyperbolic 3-manifolds.

According to Gabai, Meyerhoff and Milley,

Theorem

The unique closed minimal-volume orientable hyperbolic 3-manifold is the Weeks manifold, obtained by 5/1 and 5/2 surgery on the Whitehead link.

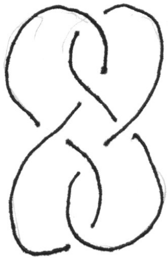
Small hyperbolic 3-manifolds

We now turn to applications to the orderability of fundamental groups of certain minimal-volume orientable hyperbolic 3-manifolds.

According to Gabai, Meyerhoff and Milley,

Theorem

The unique closed minimal-volume orientable hyperbolic 3-manifold is the Weeks manifold, obtained by 5/1 and 5/2 surgery on the Whitehead link.



Small hyperbolic 3-manifolds

Calegary and Dunfield showed that

Proposition

The fundamental group of the Weeks manifold is not left-orderable.

In fact they showed the stronger result that it is not circularly orderable.

Small hyperbolic 3-manifolds

Cao and Meyerhoff showed that there are two minimal examples in the case of one cusp.

Theorem

An orientable hyperbolic 3-manifold with one cusp, of minimal volume, is homeomorphic either to the figure 8 complement, or its sibling which is obtainable from the Whitehead link by 5/1 surgery on one component.

Small hyperbolic 3-manifolds

Cao and Meyerhoff showed that there are two minimal examples in the case of one cusp.

Theorem

An orientable hyperbolic 3-manifold with one cusp, of minimal volume, is homeomorphic either to the figure 8 complement, or its sibling which is obtainable from the Whitehead link by 5/1 surgery on one component.

Proposition

The fundamental group of the figure 8 complement is bi-orderable. That of its sibling is not bi-orderable.

Small hyperbolic 3-manifolds

Cao and Meyerhoff showed that there are two minimal examples in the case of one cusp.

Theorem

An orientable hyperbolic 3-manifold with one cusp, of minimal volume, is homeomorphic either to the figure 8 complement, or its sibling which is obtainable from the Whitehead link by 5/1 surgery on one component.

Proposition

The fundamental group of the figure 8 complement is bi-orderable. That of its sibling is not bi-orderable.

Both these examples are realized as fibrations over the circle. For the figure 8 complement, there are two positive eigenvalues. For the sibling, the eigenvalues are both negative.

Small hyperbolic 3-manifolds

Now we consider the case of two cusps.

Small hyperbolic 3-manifolds

Now we consider the case of two cusps. Agol proved the following

Theorem

A minimal volume 2-cusped orientable hyperbolic 3-manifold is homeomorphic with either the Whitehead link complement or the complement of the $(-2, 3, 8)$ *pretzel link.*

Small hyperbolic 3-manifolds

Now we consider the case of two cusps. Agol proved the following

Theorem

A minimal volume 2-cusped orientable hyperbolic 3-manifold is homeomorphic with either the Whitehead link complement or the complement of the $(-2, 3, 8)$ *pretzel link.*

Proposition

The fundamental group of the Whitehead link complement is bi-orderable.

Small hyperbolic 3-manifolds

Now we consider the case of two cusps. Agol proved the following

Theorem

A minimal volume 2-cusped orientable hyperbolic 3-manifold is homeomorphic with either the Whitehead link complement or the complement of the $(-2, 3, 8)$ *pretzel link.*

Proposition

The fundamental group of the Whitehead link complement is bi-orderable.

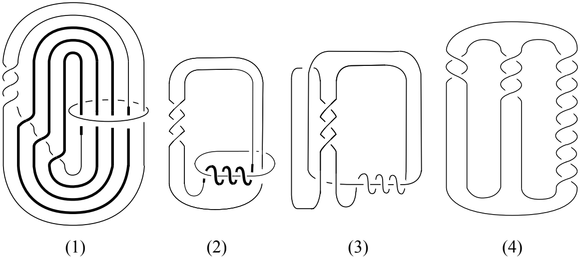
We'll use our braid ordering theory to argue that

Proposition

The fundamental group of the $(-2, 3, 8)$ *pretzel link complement is NOT bi-orderable.*

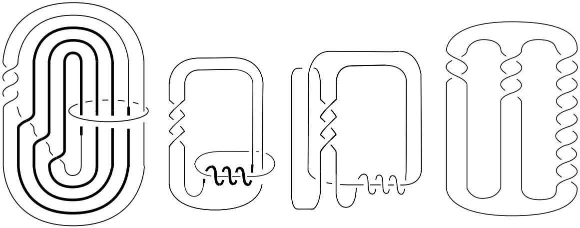
Small hyperbolic 3-manifolds

The following shows that the 5-braid $\beta = \sigma_1^3\sigma_2\sigma_3\sigma_4$ has $\mathrm{br}(\beta)$ equivalent to the (-2, 3, 8) pretzel link.



Small hyperbolic 3-manifolds

The following shows that the 5-braid $\beta = \sigma_1^3\sigma_2\sigma_3\sigma_4$ has $\mathrm{br}(\beta)$ equivalent to the (-2, 3, 8) pretzel link.



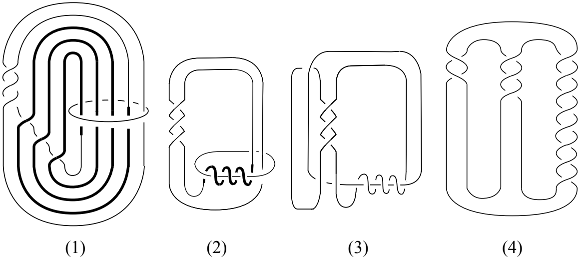
(1) (2) (3) (4)

Proposition

The braid $\beta = \sigma_1^3\sigma_2\sigma_3\sigma_4$ is not order-preserving.

Small hyperbolic 3-manifolds

The following shows that the 5-braid $\beta = \sigma_1^3\sigma_2\sigma_3\sigma_4$ has $\mathrm{br}(\beta)$ equivalent to the (-2, 3, 8) pretzel link.



Proposition

The braid $\beta = \sigma_1^3\sigma_2\sigma_3\sigma_4$ *is not order-preserving.*

This can be shown by contradiction, as in the case of the Artin generator.

Small hyperbolic 3-manifolds

Recall that $\sigma_1\sigma_2\sigma_3\sigma_4\sigma_1$ is order-preserving, and therefore its conjugate $\sigma_1^2\sigma_2\sigma_3\sigma_4$ is also order-preserving. However $\sigma_1\sigma_2\sigma_3\sigma_4$ and $\sigma_1^3\sigma_2\sigma_3\sigma_4$ are not.

Small hyperbolic 3-manifolds

Consider the following links in S^4:

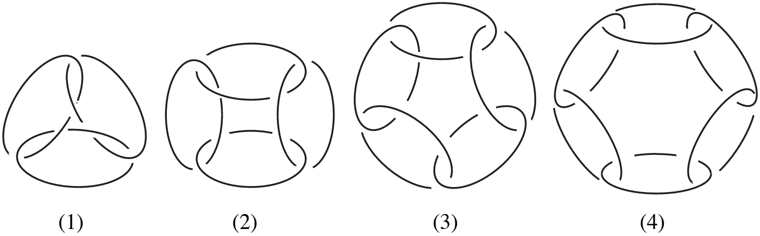


Figure: (1) C_3. (2) C_4. (3) C_5. (4) C_6.

Yoshida proved:

Theorem

A minimal volume 4-cusped orientable hyperbolic 3-manifold is homeomorphiic with $S^3 \setminus C_4$*.*

Small hyperbolic 3-manifolds

Proposition

The fundamental group of $S^3 \setminus C_4$ *is bi-orderable.*

Small hyperbolic 3-manifolds

Proposition

The fundamental group of $S^3 \setminus C_4$ is bi-orderable.

This follows since the complement of C_4 is homeomorphic with the complement of $\mathrm{br}(\sigma_1^{-2}\sigma_2^2)$, whose group is biorderable, since the braid is a pure braid.

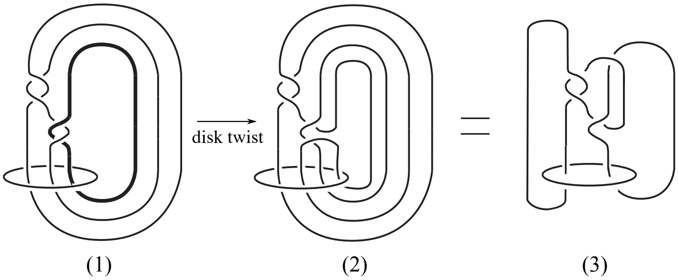


Figure: $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^2)$ is homeomorphic to $S^3 \setminus C_4$. (1) $\mathrm{br}(\sigma_1^{-2}\sigma_2^2)$. (2)(3) Links which are equivalent to C_4.

Small hyperbolic 3-manifolds

It has been conjectured that the complement of C_5 has volume smaller than all other 5-cusped hyperbolic orientable 3-manifolds.

Small hyperbolic 3-manifolds

It has been conjectured that the complement of C_5 has volume smaller than all other 5-cusped hyperbolic orientable 3-manifolds.

Proposition

The fundamental group of $S^3 \setminus C_5$ *is bi-orderable.*

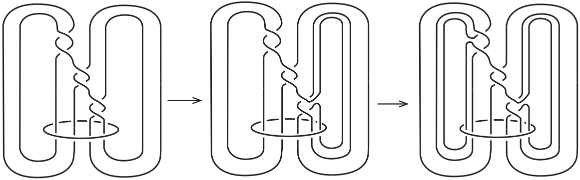
Small hyperbolic 3-manifolds

It has been conjectured that the complement of C_5 has volume smaller than all other 5-cusped hyperbolic orientable 3-manifolds.

Proposition

The fundamental group of $S^3 \setminus C_5$ is bi-orderable.

This follows because it is homeomorphic with $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$



(1) (2) (3)

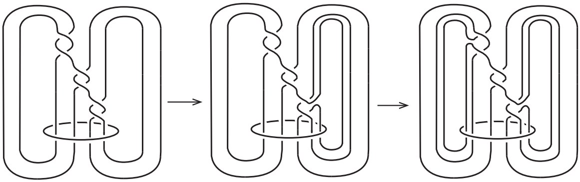
Small hyperbolic 3-manifolds

It has been conjectured that the complement of C_5 has volume smaller than all other 5-cusped hyperbolic orientable 3-manifolds.

Proposition

The fundamental group of $S^3 \setminus C_5$ is bi-orderable.

This follows because it is homeomorphic with $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2})$



(1) (2) (3)

Similarly, $S^3 \setminus C_6$ has bi-orderable fundamental group, as it is homeomorphic to $S^3 \setminus \mathrm{br}(\sigma_1^{-2}\sigma_2^{-2}\sigma_3^{-2}\sigma_4^{-2})$.

Small hyperbolic 3-manifolds

We'll end with the case of 3 cusps. $S^3 \setminus C_3$ is known as the "Magic manifold," as it was called by Gordon and Wu. It is conjectured to be the minimal volume 3-cusped orientable hyperbolic 3-manifold.

Small hyperbolic 3-manifolds

We'll end with the case of 3 cusps. $S^3 \setminus C_3$ is known as the "Magic manifold," as it was called by Gordon and Wu. It is conjectured to be the minimal volume 3-cusped orientable hyperbolic 3-manifold.

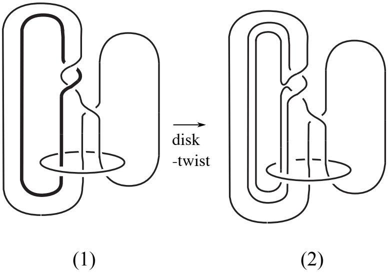


Figure: $S^3 \setminus \mathrm{br}(\sigma_1^2\sigma_2^{-1})$ is homeomorphic to $S^3 \setminus C_3$. (1) $\mathrm{br}(\sigma_1^2\sigma_2^{-1})$. (2) Link which is equivalent to C_3.

Small hyperbolic 3-manifolds

Question: Is the fundamental group of the magic manifold bi-orderable? Equivalently, is the 3-braid $\sigma_1^2\sigma_2^{-1}$ order-preserving?

Small hyperbolic 3-manifolds

Question: Is the fundamental group of the magic manifold bi-orderable? Equivalently, is the 3-braid $\sigma_1^2\sigma_2^{-1}$ order-preserving?

THANK YOU