Derived equivalences for skew-gentle algebras.

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Aim : combine the following two constructions.

1. from Λ an algebra acted on by a group G, we can define a new algebra ΛG, and nice functors linking the representations of Λ and the representations of ΛG [Reiten-Riedtmann 85]
2. to Λ a gentle algebra, one can associate a marked surface with a collection of arcs [Opper-Plamondon-Schroll 2018], and algebraic properties of $\mathcal{D}^b(\Lambda)$ can be interpreted using the geometry [A-Plamondon-Schroll, Opper 2019].

Question

Let Λ and Λ' be gentle algebras with a certain action of a group G. Can we find a geometric interpretation of the fact that ΛG and $\Lambda' G$ have the same derived category ?

Notation : k field, G finite abelian group such that $|G|$ invertible in k, Λ finite dimensional k-algebra with a G-action by automorphism.

We define ΛG as

$$\Lambda G = \Lambda \otimes kG \textit{ and } (\lambda \otimes g).(\lambda' \otimes g') = \lambda g(\lambda') \otimes gg'.$$

Since ΛG is a natural left Λ-module, we get adjoint functors

$$\begin{tikzcd} \mathcal{D}^b(\Lambda) \arrow[r, shift left, "-\overset{\mathbf{L}}{\underset{\Lambda}{\otimes}}\Lambda G"] & \mathcal{D}^b(\Lambda G) \arrow[l, shift left, "Res"] \end{tikzcd}$$

Let $\widehat{G} = \mathrm{Hom}(G, k^*)$ be the dual group. Then $\widehat{G}$ acts on ΛG by

$$\chi.(\lambda \otimes g) = \chi(g)\lambda \otimes g$$

Proposition (RR'85)

The algebras $(\Lambda G)\widehat{G}$ and Λ are Morita equivalent.

Example

Let $\Lambda = k$ with trivial action of $G = \mathbb{Z}/2\mathbb{Z}$.
Then $\Lambda G = k \times k$. The action of $\widehat{G}$ exchanges the two copies of k.
$\Lambda G \widehat{G} = \mathrm{Mat}_2(k)$. It is Morita equivalent to k.

Example

$$\Lambda = \begin{tikzcd} & 2^+ \arrow[r] & 3^+ \arrow[r] & 4^+ \\ 1 \arrow[ur] \arrow[dr] & & & \\ & 2^- \arrow[r] & 3^- \arrow[r] & 4^- \end{tikzcd}$$

$$\Lambda G \sim \begin{tikzcd} 1_0 \arrow[dr] & & & \\ & 2 \arrow[r] & 3 \arrow[r] & 4 \\ 1_1 \arrow[ur] & & & \end{tikzcd}$$

Definition

An object $T \in \mathcal{D}^b(\Lambda)$ is called *tilting* if

$$\forall i \neq 0,\ \mathrm{Ext}^i(T,T) = 0 \quad \text{and} \quad \mathrm{thick}(T) = \mathcal{D}^b(\Lambda).$$

Theorem (Happel-Rickard)

Let Λ and Λ' be finite dimensional algebras. Then $\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\Lambda')$ if and only if there exists a tilting object $T \in \mathcal{D}^b(\Lambda)$ such that $\mathrm{End}(T) \simeq \Lambda$.

Fact : If $T \in \mathcal{D}^b(\Lambda)$ is G-invariant, then $\mathrm{End}(T)$ has a natural G-action.

Theorem (A-Brüstle)

Let Λ and Λ' be algebras with G-actions, then we have

$$\mathcal{D}^b(\Lambda) \underset{G}{\sim} \mathcal{D}^b(\Lambda') \Rightarrow \mathcal{D}^b(\Lambda G) \underset{\widehat{G}}{\sim} \mathcal{D}^b(\Lambda' G).$$

If T is tilting G-invariant, then $T \overset{\mathbf{L}}{\underset{\Lambda}{\otimes}} \Lambda G$ is tilting $\widehat{G}$-invariant.

Remark

$$\mathcal{D}^b(\Lambda G) \underset{\widehat{G}}{\sim} \mathcal{D}^b(\Lambda' G) \Rightarrow \mathcal{D}^b(\Lambda G \widehat{G}) \underset{G}{\sim} \mathcal{D}^b(\Lambda' G \widehat{G}) \Rightarrow \mathcal{D}^b(\Lambda) \sim \mathcal{D}^b(\Lambda').$$

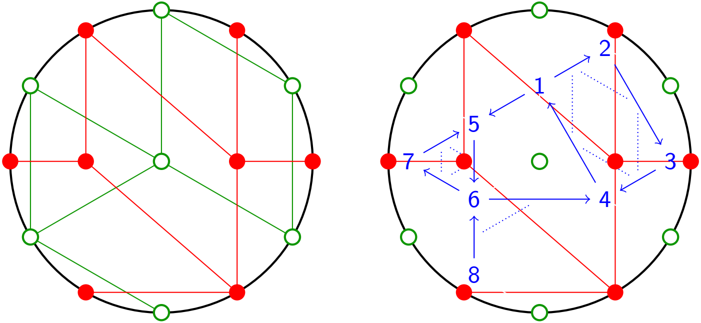
But it is not clear that it implies $\mathcal{D}^b(\Lambda) \underset{G}{\sim} \mathcal{D}^b(\Lambda')$.

Let (S, M, P) be a marked surface. $M \subset \partial S$. A **dissection** D on (S, M, P) is a maximal collection of non intersecting arcs with endpoints in M or P, that do not cut out a subsurface of S.

To (S, M, P, D), one can associate an algebra $\Lambda = kQ/I$ which is called **gentle** (cf Pierre-Guy Plamondon's talk).

Let $\sigma \in \mathrm{Homeo}^+(S)$ of order 2 with finitely many fiwed points such that $\sigma(M) = M$, $\sigma(P) = P$ and $\sigma(D) = D$. This defines a $\mathbb{Z}/2\mathbb{Z}$-action on Λ.

Aim : Give a geometric model for the algebras ΛG.


1
2
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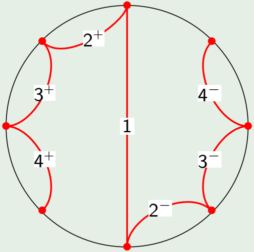
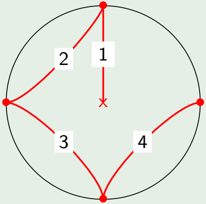
Example

(S, M)



$$\begin{tikzcd}[row sep=small] & 2^+ \arrow[r] & 3^+ \arrow[r] & 4^+ \\ 1 \arrow[ur] \arrow[dr] & & & \\ & 2^- \arrow[r] & 3^- \arrow[r] & 4^- \end{tikzcd}$$

$(\bar{S}, \bar{M}) = (S, M)/\sigma$



$$\begin{tikzcd}[row sep=small] 1_0 \arrow[dr] & & & \\ & 2 \arrow[r] & 3 \arrow[r] & 4 \\ 1_1 \arrow[ur] & & & \end{tikzcd}$$

Proposition (AB)

ΛG is a skew-gentle algebra. All skew-gentle algebras are obtained in this way.

Skew-gentle algebras :[Geiss-de la Peña '95]. contains all gentle algebras, and D_n, $\widetilde{D}_n$ quivers.

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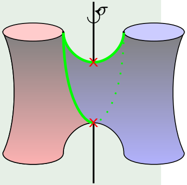
$$\begin{tikzcd}
& 2_0 \arrow[r] \arrow[ddr] & 3_0 \arrow[r] \arrow[ddr] & 4_0 \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & & & 5 \\
& 2_1 \arrow[r] \arrow[uur] & 3_1 \arrow[r] \arrow[uur] & 4_1 \arrow[ur] &
\end{tikzcd}$$

But, if A is skew-gentle, then the (S, M, D) is not unique.

Example

$\chi_1(\alpha) = \alpha$

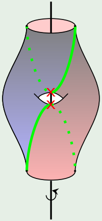
$$\Lambda_1 = \begin{tikzcd} 1^+ \arrow[rrr] \arrow[dr] & & & 4^+ \\ & 2 \arrow[r, shift left] \arrow[r, shift right] & 3 \arrow[ur] \arrow[dr] & \\ 1^- \arrow[rrr] \arrow[ur] & & & 4^- \end{tikzcd}$$



$$A = \begin{tikzcd} & 2_0 \arrow[r] \arrow[dr] & 3_0 \arrow[dr] & \\ 1 \arrow[ur] \arrow[dr] \arrow[rrr, "\alpha"] & & & 4 \\ & 2_1 \arrow[r] \arrow[ur] & 3_1 \arrow[ur] & \end{tikzcd}$$

$\chi_2(\alpha) = -\alpha$

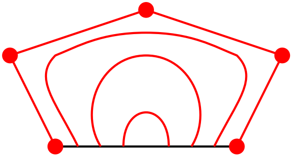
$$\Lambda_2 = \begin{tikzcd} 1^+ \arrow[dr] \arrow[rrrdd] & & & 4^+ \\ & 2 \arrow[r, shift left] \arrow[r, shift right] & 3 \arrow[ur] \arrow[dr] & \\ 1^- \arrow[ur] \arrow[rrruu] & & & 4^- \end{tikzcd}$$



Theorem (AB'19)

Let A and A' be two skew-gentle algebras, and Λ and Λ' the corresponding G-gentle algebras. Then the following are equivalent :

1. $\mathcal{D}^b(A) \underset{\widehat{G}}{\sim} \mathcal{D}^b(A')$.
2. $\mathcal{D}^b(\Lambda) \underset{G}{\sim} \mathcal{D}^b(\Lambda')$
3. *there exists a G-homeomorphism $(S, M, \eta) \to (S', M', \eta')$.*



Theorem (AB'19)

Let A and A' be two skew-gentle algebras. Then the following are equivalent :

1. *there exists a $\widehat{G}$-invariant tilting object T in $\mathcal{D}^b(A)$ with $\mathrm{End}(T) \simeq A'$;*
2. *there exists a homeomorphism $(\bar{S}, \bar{M}, \bar{\eta}) \to (\bar{S}', \bar{M}', \bar{\eta}')$.*

Here $\bar{S}$ is the orbifold S/σ.

$(\bar{S}, \bar{D})$	A	(S, σ)	Λ