Computations in the geometric Satake correspondence

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partially based on joint work with S. Gaussent and P. Littelmann

Plan of the talk

- The dual canonical basis for $\mathbf{SL}_2$
- Geometric Satake correspondence and Mirković-Vilonen bases
- Comparison with the dual canonical bases

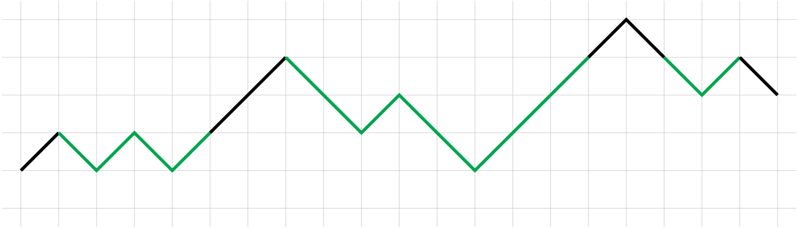
The dual canonical basis for $\mathbf{SL}_2$

$V = \mathbb{C}^2$ natural representation of $\mathbf{SL}_2$ with basis (x_+, x_-)

$\mathcal{C}_n = \{+,-\}^n$ set of words

Graphical representation of

$$+-+-+++--+--++++--+-$$



Any word can be obtained by inserting repetitively $-+$ in a word of the form $++\cdots+--\cdots-$

Tensor product basis of $V^{\otimes n}$:

$x_w = x_{w(1)} \otimes \cdots \otimes x_{w(n)}$ for $w \in \mathcal{C}_n$

Recursive procedure to define a new basis $(y_w)_{w \in \mathcal{C}_n}$:

$$\boxed{y_w = x_w \ \text{ if } w \text{ is of the form } ++\cdots+--\cdots-}$$

for $w = w' -+ w''$ with $(w', w'') \in \mathcal{C}_{n'} \times \mathcal{C}_{n''}$

write $y_{w'w''} = \sum_i y_i' \otimes y_i''$ with $(y_i', y_i'') \in V^{\otimes n'} \times V^{\otimes n''}$

$$\boxed{y_w = \sum_i y_i' \otimes y_{-+} \otimes y_i''}$$

where $y_{-+} = x_{-+} - x_{+-}$ ($\mathbf{SL}_2$-invariant in $V^{\otimes 2}$)

Theorem (Frenkel-Khovanov, '97):

The elements y_w are well-defined and $(y_w)_{w\in\mathcal{C}_n}$ is the dual canonical basis of $V^{\otimes n}$ (at $q=1$).

Transition matrix

$$x_w = y_w + \sum_{w' \text{ above } w} n_{w,w'}\, y_{w'} \quad \text{where } n_{w,w'} \in \mathbb{Z}_{\geq 0}$$

Compatibility with the isotypical filtration of $V^{\otimes n}$

e, f, h = the usual elements in $\mathfrak{sl}_2$

$\tilde{e}$: replaces the leftmost black $-$ in a word by a (black) $+$

$\tilde{f}$: replaces the rightmost black $+$ in a word by a (black) $-$

$\ell(w) = \#\{\text{black letters in a word } w\}$

$(V^{\otimes n})_{\leq p} = \text{span}\{y_w \mid w \in \mathcal{C}_n,\ \ell(w) \leq p\}$

$$e \cdot y_w \equiv y_{\tilde{e}(w)} \mod (V^{\otimes n})_{\leq \ell(w)-2}$$

$$f \cdot y_w \equiv y_{\tilde{f}(w)} \mod (V^{\otimes n})_{\leq \ell(w)-2}$$

Example: $\{y_w \mid \ell(w) = 0\}$ is a basis of the invariants $(V^{\otimes n})^{\mathbf{SL}_2}$.

Generalization

Can one extend the construction to tensor products

$$V(\underline{\lambda}) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$$

of irreducible (f.d.) representations of a reductive group G?

First possibility: endow each $V(\lambda_i)$ with its dual canonical basis (upper global basis) and modify the tensor product of these bases to define the dual canonical basis of $V(\underline{\lambda})$.

Again (Lusztig, '93):

- Unitriangular transition matrix;
- Compatibility with the isotypical filtration;
- The Chevalley generators of $\mathfrak{g}$ act on the dual canonical basis of $V(\underline{\lambda})$ compatibly with a crystal structure.

Geometric Satake correspondence

G connected reductive group over $\mathbb{C}$

$T \subset B$ maximal torus and Borel subgroup

N^- unipotent radical of opposite Borel

$\Lambda = X_*(T) \supset \Lambda^+$ cocharacter lattice and dominant cone

$G^\vee \supset B^\vee \supset T^\vee$ Langlands dual; thus $X^*(T^\vee) = \Lambda$

$\mathcal{O} = \mathcal{C}[[t]], \ \mathcal{K} = \mathbb{C}((t))$

$\mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O})$ affine Grassmannian of G

$\lambda \in \Lambda \rightsquigarrow \lambda(t) \in T(\mathcal{K}) \rightsquigarrow L_\lambda \in \mathsf{Gr}$

For $\lambda \in \Lambda^+$, define

$\mathsf{Gr}^\lambda = G(\mathcal{O}) \cdot L_\lambda$

$V(\lambda)$ = the irreducible $G^\vee$-module of h.w. λ

Geometric Satake correspondence:

$\mathsf{IH}(\overline{\mathsf{Gr}^\lambda}, \mathbb{C}) \cong V(\lambda)$ (Lusztig)

+ compatibility with $\otimes$ (Beilinson-Drinfeld, Mirković-Vilonen)

+ action of $G^\vee$ (Ginzburg, Vasserot)

+ bases (MV, Goncharov-Shen)

+ crystal structure (Braverman-Gaitsgory)

(+ applications. . .)

Tensor products

$$\begin{tikzcd} \mathrm{Gr}_n = \underbrace{G(\mathcal{K}) \times^{G(\mathcal{O})} G(\mathcal{K}) \cdots \times^{G(\mathcal{O})} G(\mathcal{K})}_{n \text{ factors}} / G(\mathcal{O}) \arrow[d, "m"'] \\ \mathrm{Gr} = G(\mathcal{K})/G(\mathcal{O}) \end{tikzcd}$$

where $m([g_1, \ldots, g_n]) = [g_1 \cdots g_n]$.

For $\underline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\Lambda^+)^n$, define

$$\mathrm{Gr}_n^{\underline{\lambda}} = \widehat{\mathrm{Gr}^{\lambda_1}} \times^{G(\mathcal{O})} \widehat{\mathrm{Gr}^{\lambda_2}} \cdots \times^{G(\mathcal{O})} \widehat{\mathrm{Gr}^{\lambda_n}} / G(\mathcal{O})$$

where $\widehat{\cdot}$ means preimage under $G(\mathcal{K}) \to \mathrm{Gr}$.

Geometric Satake: $\mathrm{IH}(\overline{\mathrm{Gr}_n^{\underline{\lambda}}}, \mathbb{C}) \cong V(\underline{\lambda}) = V(\lambda_1) \otimes \cdots \otimes V(\lambda_n)$.

The Mirković-Vilonen basis

$\theta \in X^*(T)$ regular dominant $\rightsquigarrow$

$$\text{action} \quad \mathbb{C}^\times \xrightarrow{\theta} T(\mathbb{C}) \to G(\mathcal{K}) \curvearrowright \mathrm{Gr}$$

$\mu \in \Lambda \rightsquigarrow$ unstable set $T_\mu = \left\{ x \in \mathrm{Gr} \;\middle|\; \lim_{c\to\infty} \theta(c)\cdot x = L_\mu \right\}$

$$= N^-(\mathcal{K}) \cdot L_\mu \quad \text{(Iwasawa decomposition)}$$

Theorem (Mirković-Vilonen '00, Goncharov-Shen '15):

$$\mathrm{IH}(\overline{\mathrm{Gr}^{\underline{\lambda}}_n}, \mathbb{C}) = \bigoplus_{\mu\in\Lambda} \underbrace{\mathrm{H}_{m^{-1}(T_\mu)}(\mathrm{IC}(\overline{\mathrm{Gr}^{\underline{\lambda}}_n}, \mathbb{C}))}$$

$$\downarrow \simeq$$

$$\mathrm{H}_{\mathrm{top}}(\overline{\mathrm{Gr}^{\underline{\lambda}}_n} \cap m^{-1}(T_\mu))$$

The MV basis of $V(\underline{\lambda})_\mu$ is formed by the fundamental classes of the irreducible components of $\overline{\mathrm{Gr}^{\underline{\lambda}}_n} \cap m^{-1}(T_\mu)$.

Notation: $\mathcal{Z}(\underline{\lambda})_\mu = \operatorname{Irr}(\overline{\mathrm{Gr}^{\underline{\lambda}}_n} \cap m^{-1}(T_\mu))$ (MV cycles)

How to compute in the MV basis?

- Choose a projective embedding of Gr: the principal nilpotent of $\mathfrak{g}^\vee$ acts in the MV basis of $V(\lambda)$ by cutting a cycle $Z \in \mathcal{Z}(\lambda)_\mu$ by an hyperplane (Ginzburg, Vasserot).
- The Geometric Satake correspondence is built atop an equivalence of Tannakian categories. One needs the Beilinson-Drinfeld Grassmannian to prove that the fiber functor is monoidal.
- MV cycles can be described in coordinates with Littelmann's path model (Gaussent-Littelmann; also Ngô-Polo).

Theorem (B-Gaussent-Littelmann):

- The Chevalley generators of $\mathfrak{g}^\vee$ act on the MV basis of $V(\underline{\lambda})$ compatibly with the Braverman-Gaitsgory crystal structure on $\bigsqcup_{\mu\in\Lambda} \mathcal{Z}(\underline{\lambda})_\mu$.
- The MV basis of $V(\underline{\lambda})$ is compatible with the isotypical filtration of this module.
- The signed cyclic permutation $$(V(\lambda_1)\otimes\cdots\otimes V(\lambda_n))^{G^\vee} \to (V(\lambda_2)\otimes\cdots\otimes V(\lambda_n)\otimes V(\lambda_1))^{G^\vee}$$ maps the MV basis on the left to the MV basis on the right.
- The MV basis of $V(\underline{\lambda})$ induces the MV basis at the top step $V(\lambda_1+\cdots+\lambda_n)$ of the isotypical filtration.

Comparison of bases

Given $N^-(\mathcal{O})$-stable subsets $Z_1 \subseteq T_{\mu_1}, \ldots, Z_n \subseteq T_{\mu_n}$, define

$$Z_1 \ltimes \cdots \ltimes Z_n = \widetilde{Z}_1 \times^{N^-(\mathcal{O})} \widetilde{Z}_2 \cdots \times^{N^-(\mathcal{O})} \widetilde{Z}_n \,/\, N^-(\mathcal{O})$$

where $\widetilde{Z}_j$ is the preimage of Z_j under $N^-(\mathcal{K})\, t^{\mu_j} \to T_{\mu_j}$.

There is a natural bijective map

$$\Upsilon_\mu : \bigsqcup_{\mu_1+\cdots+\mu_n=\mu} T_{\mu_1} \ltimes \cdots \ltimes T_{\mu_n} \to T_\mu .$$

If $(Z_1, \ldots, Z_n) \in \mathcal{Z}(\lambda_1)_{\mu_1} \times \cdots \times \mathcal{Z}(\lambda_n)_{\mu_n}$, then

$$\overline{\Upsilon_\mu(Z_1 \ltimes \cdots \ltimes Z_n)} \in \mathcal{Z}(\underline{\lambda})_\mu \quad \text{where } \mu = \mu_1 + \cdots + \mu_n .$$

$\Rightarrow$ bijection $\displaystyle\bigsqcup_{\mu_1+\cdots+\mu_n=\mu} \mathcal{Z}(\lambda_1)_{\mu_1} \times \cdots \times \mathcal{Z}(\lambda_n)_{\mu_n} \cong \mathcal{Z}(\underline{\lambda})_\mu .$

Theorem (BGL): The resulting bijection

$$\mathcal{Z}(\underline{\lambda}) \cong \mathcal{Z}(\lambda_1) \times \cdots \times \mathcal{Z}(\lambda_n).$$

is a crystal isomorphism.

For $Z = \overline{\Upsilon_\mu(Z_1 \ltimes \ldots \ltimes Z_n)}$, define

- $y(Z) = [Z]$ an element in the MV basis of $V(\underline{\lambda})$.
- $x(Z) = [Z_1] \otimes \cdots \otimes [Z_n]$ an element in the tensor product of the MV bases of the factors $V(\lambda_j)$.

Transition matrix

$$x(Z) = y(Z) + \sum_{Z' \neq Z} n(Z, Z')\, y(Z').$$

Theorem (BGL):

1. $n(Z, Z') \in \mathbb{Z}_{\geq 0}$.
2. Write $(\mu_1, \ldots, \mu_n)$ for the weights of $(Z_1, \ldots, Z_n)$ and $(\mu'_1, \ldots, \mu'_n)$ for the weights of $(Z'_1, \ldots, Z'_n)$.

$$n(Z, Z') > 0 \;\Rightarrow\; \begin{aligned} &\mu'_1 \geq \mu_1 \\ &\mu'_1 + \mu'_2 \geq \mu_1 + \mu_2 \\ &\qquad\qquad \ddots \\ &\mu'_1 + \cdots + \mu'_n = \mu_1 + \cdots + \mu_n \end{aligned}$$

with at least one strict inequality.

Using Beilinson and Drinfeld's fusion product, one can actually compute $n(Z, Z')$: they are intersection multiplicities.

Case $G^\vee = \mathbf{SL}_2$:

Here the MV basis and the dual canonical basis of $V(\underline{\lambda})$ coincide (Demarais).

Wrong in general!

First counterexample: for $G^\vee = \mathbf{SL}_3$, Fontaine-Kamnitzer-Kuperberg ('13) give a counterexample in the $\mathbf{SL}_3$-invariants in $V(\underline{\lambda})$ with

$$\underline{\lambda} = (\varpi_1, \varpi_2, \varpi_2, \varpi_1, \varpi_1, \varpi_2, \varpi_2, \varpi_1, \varpi_1, \varpi_2, \varpi_2, \varpi_1).$$

Second counterexample: consider $G^\vee = \mathbf{SO}_8$ and $\lambda = 2\varpi_2$ (ϖ_2 is the highest root).

Given b in the crystal $B(\lambda)$, denote:

$G^*(b)$ the element in the dual canonical (upper global) basis;

$S^*(b)$ the element in the dual semicanonical basis;

$M^*(b)$ the element in the MV basis.

Set $b_1 = (\tilde{f}_2\tilde{f}_1\tilde{f}_3\tilde{f}_4\tilde{f}_2)^2 b_{\mathrm{top}}$ and $b_2 = (\tilde{f}_2)^2(\tilde{f}_1\tilde{f}_3\tilde{f}_4)^2(\tilde{f}_2)^2 b_{\mathrm{top}}$.

Geiß-Leclerc-Schröer ('05):
$$S^*(b_1) = G^*(b_1),$$
$$S^*(b_2) = G^*(b_2) + G^*(b_1).$$

In addition:
$$M^*(b_1) = G^*(b_1),$$
$$M^*(b_2) = G^*(b_2) - G^*(b_1).$$

Sketch of proof

Define two elements

$$b_3 = (\tilde{f}_1\tilde{f}_3\tilde{f}_4\tilde{f}_2)b_{\text{top}} \quad \text{and} \quad b_4 = ((\tilde{f}_2)^2\tilde{f}_1\tilde{f}_3\tilde{f}_4\tilde{f}_2)b_{\text{top}}$$

in $B(\varpi_2)$. By computing the relevant intersection multiplicities

$$M^*(b_3) \otimes M^*(b_4) = 2M^*(b_1) + M^*(b_2) + (\text{other terms}).$$

Complete the argument by projecting onto the Cartan component of the tensor square $V(\varpi_2)^{\otimes 2}$.

(Note that $M^*(b_3) = G^*(b_3)$ and $M^*(b_4) = G^*(b_4)$ in $V(\varpi_2)$ since these are just root vectors.)

A basis of $\mathbb{C}[N^\vee]$

$N^\vee$ unipotent radical of $B^\vee$

$\Psi_\lambda : V(\lambda) \hookrightarrow \mathbb{C}[N^\vee]$ given by $\Psi_\lambda(v) = (u \mapsto (v_\lambda, u \cdot v))$

where $v_\lambda \in V(\lambda)$ h.w. vector and $(\ ,\ )$ contravariant form.

Fact: Through Ψ_λ, the MV bases of the various $V(\lambda)$ match

$\rightsquigarrow$ the MV basis of $\mathbb{C}[N^\vee]$

has properties similar to the dual semicanonical and dual canonical (upper global) bases.

Question: Do the cluster monomials of $\mathbb{C}[N^\vee]$ belong to this basis?

Not known in general.

A very rudimentary result

$(s_i)_{i \in I}$ the simple reflections

$(\varpi_i)_{i \in I}$ the fundamental weights

$C \subset \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ the dominant Weyl chamber

Choose a reduced word $\underline{i} = (i_1, \ldots, i_\ell)$, set $w_k = s_{i_1} \cdots s_{i_k}$,

and say that $\underline{i}$ satisfies (A) if

$$\boxed{\begin{array}{l} \text{There exist } \nu_1 \in w_1(C), \ \ldots, \ \nu_\ell \in w_\ell(C) \\ \text{such that } \nu_k - \nu_{k+1} \in C \text{ for all } k \in \{1, \ldots, \ell - 1\}. \end{array}}$$

Denote by $v_{w\lambda}$ the extremal weight vector in $V(\lambda)$.

Proposition:

Let $f_1, \ldots, f_\ell \in \mathbb{C}[N^\vee]$ be the flag minors along $\underline{i}$

$$f_j = \Psi_{\varpi_{i_j}}(v_{w_j \varpi_{i_j}}).$$

If $\underline{i}$ satisfies (A), then any monomial in f_1, ..., f_ℓ belongs to the MV basis of $\mathbb{C}[N^\vee]$.

However any interpretation of the exchange relation in the context of Geometric Satake remains elusive.

Merci Bernard

et joyeux anniversaire !