# Macdonald Polynomials and Character Formulae.

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March 2020

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In doing this, they identified certain interesting families of irreducible representations.

The cluster variables correspond to what is called a prime real representation and the cluster monomials to irreducible tensor products of such representations.

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I want to explain the connection between this decomposition and Macdonald polynomials in type $A_n$.

I would also to explain the connections of this character formula with Macdonald polynomials in type $B_n$.

# The category $\widehat{\mathcal{F}}_q$

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Suppose $\boldsymbol{\omega} \in \mathcal{P}^+$; then we can write it as a product

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k}, \quad 1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$$

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for some choice of parameters. It is clear that given $\boldsymbol{\omega}$ there is an associated partition whose Young diagram is given by $k$ columns of height $i_1, \cdots, i_k$ respectively.

# The category $\mathcal{F}_q$

So we have a map $\mathcal{P}^+$ to $P^+$ (dominant integral weights or partitions with at most $n$ rows)

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k} \rightsquigarrow \lambda = \omega_{i_1} + \cdots + \omega_{i_k}.$$

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In the most generic case ($a_i/a_j \notin q^{\mathbb{Z}}$) one knows that the character of the irreducible representation is just the product of the characters of $[\boldsymbol{\omega}_{i,a}]$ and these are known. They are just the characters of the fundamental modules for $\mathfrak{sl}_{n+1}$.

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But in the non-generic case this problem is hard and known, only in certain special cases, for instance the evaluation modules $V(\lambda)$ and certain other cases which are usually suitable tensor products of these.

# Prime Representations

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But the problem, of even finding large classes of examples of prime objects, leave alone classifying them, seems very hard.

And this is where the approach through monoidal categorification has been very helpful.

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• a single column; the associated module is the irreducible module with index $\boldsymbol{\omega}_{i,a}$; usually part of the initial seed,

• two columns, both of the same height say $i$; "frozen variables"; the associated HL-module is the irreducible module with index $\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{i,aq^2}$,

• all the columns have distinct heights. "unfrozen cluster variables". The associated module is indexed by $\boldsymbol{\omega}_{i_1,a_1}\cdots\boldsymbol{\omega}_{i_k,a_k}$ with $i_1 < \cdots < i_k$ and $a_1, \cdots, a_k$ depend on the difference of column heights as follows:

$$a_1 = 1, \quad a_2 = q^{i_2 - i_1 + 2}, \quad a_3 = q^{-i_3 + 2i_2 - i_1} \cdots \ldots.$$

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The character in the case when the partition has one or two columns is easy. It is just the character of the corresponding irreducible module for $\mathfrak{sl}_{n+1}$.

# HL modules

In the case when the partition has three or more columns, one knows that the character of the corresponding prime module,

$$\mathrm{ch}\ \hat{V}_q(\lambda) = s_\lambda + \sum_{\mu<\lambda} r_\lambda^\mu s_\mu, \quad r_\lambda^\mu \in \mathbf{Z}_+$$

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Clearly, $\mathrm{ch}\hat{V}_q(\lambda)$ define a linearly independent family of symmetric functions.

So my goal is to explain how these characters arise as specializations at $\mathbf{q} = 1$ of a family of polynomials $G_\lambda(z, \mathbf{q})$ which in turn are defined in terms of specialized Macdonald polynomials, $P_\lambda(z, \mathbf{q}, 0)$. Since $q$ is being used for the quantum parameter, I am using $\mathbf{q}$ for the parameter which shows up in Macdonald theory!

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$\lambda = 4 \geq 2 \geq 1$ would correspond to $(\boldsymbol{\omega}_{1,1}\boldsymbol{\omega}_{1,q^2}) \otimes (\boldsymbol{\omega}_{2,1}\boldsymbol{\omega}_{3,q^3})$ (frozen tensor prime) and $G_\lambda(z, 1)$ will give the character of this module.

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$\lambda = 3 \geq 2 \geq 2 \geq 2 \geq 1$ would correspond to the prime module given by $\boldsymbol{\omega}_{1,1}\boldsymbol{\omega}_{4.q^5}\boldsymbol{\omega}_{5,q^2}$.

# The polynomials $G_\lambda(z, \mathbf{q})$

Fix a set of indeterminates $z = (z_1, \cdots, z_{n+1})$ and $\mathbf{q}$. Let $P_\lambda(z, \mathbf{q}, 0)$ be the specialized Macdonald polynomial associated to $\lambda$.

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Suppose that we have a collection of polynomials $p_\lambda^\mu(\mathbf{q}) \in \mathbb{Z}_+[\mathbf{q}]$ where $\lambda, \mu$ vary over all partitions, and satisfy:

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Then we can define polynomials $G_\lambda(z, \mathbf{q})$ (depending on $p_\lambda^\mu(q)$) recursively, by requiring

$$P_0(z, \mathbf{q}, 0) = G_0(z, \mathbf{q}) = 1, \quad P_{\omega_i}(z, \mathbf{q}, 0) = G_{\omega_i}(z, \mathbf{q}),$$

$$P_\lambda(z, \mathbf{q}, 0) = \sum_\mu p_\lambda^\mu(\mathbf{q}) G_\mu(z, \mathbf{q}).$$

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Of course, in general the $G_\lambda(z, \mathbf{q})$ are not going to be the characters

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I now want to explain that there is a particular choice of $p_\lambda^\mu(\mathbf{q})$ so that the polynomials $G_\mu(z,\mathbf{q})$ defined by these formulae,

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We will see that at $\mathbf{q} = 1$ they give the character of the HL-module. And that they are connected with Macdonald polynomials associated to non-simply laced root systems.

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Our convention is that $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathbf{q}} =$ if $m < 0$ or $m > n$.

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Our convention is that $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathbf{q}} =$ if $m < 0$ or $m > n$.

Notice that $p_\lambda^\lambda = 1, \quad p_\lambda^\mu = 0 \ \ \mu \not\leq \lambda$. Moreover,

$(\lambda+\mu_1, \lambda-\mu) = (\lambda-\mu, \lambda-\mu) + 2(\mu-\mu_0, \lambda-\mu) \in 2\mathbb{Z}_+,$ if $\lambda-\mu \in Q^+$,

and in particular $p_\lambda^\mu \in \mathbb{Z}_+[q]$.

# Example in $\mathfrak{sl}_3$.

Suppose that $\mathfrak{g} = \mathfrak{sl}_3$. If $\lambda = \omega_1 + \omega_2$ and $\mu = 0$ we have $\lambda - \mu = \alpha_1 + \alpha_2$ and so we get,

$$p^0_{\omega_1+\omega_2} = \mathbf{q} \begin{bmatrix} (\alpha_1 + \alpha_2, \omega_1) \\ (\alpha_1 + \alpha_2, \omega_1) \end{bmatrix}_{\mathbf{q}} \begin{bmatrix} (\alpha_1 + \alpha_2, \omega_2) \\ (\alpha_1 + \alpha_2, \omega_2) \end{bmatrix}_{\mathbf{q}} = \mathbf{q}.$$

Moreover,

$$P_{\omega_1+\omega_2} = s_{\omega_1+\omega_2} + \mathbf{q} = G_{\omega_1+\omega_2} + \mathbf{q} \implies G_{\omega_1+\omega_2} = s_{\omega_1+\omega_2}.$$

This is exactly what one expects in the HL-module in this case since the choice of parameters guarantees that its character is a Schur polynomial.

# Example of $G_\lambda$ in $\mathfrak{sl}_4$

The most interesting HL-module in $\mathfrak{sl}_4$ is associated with $\omega_1 + \omega_2 + \omega_3$; namely $\boldsymbol{\omega}_{1,1}\boldsymbol{\omega}_{2,q^3}\boldsymbol{\omega}_{3,1}$. After a small calculation we find that:

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At $\mathbf{q} = 1$ one can check using elementary representation theory that this is the character of the HL-module.

Already for $\mathfrak{sl}_5$ this probelem becomes hard to do by brute force.

## The main results

**Theorem[Biswal- C-Shereen-Wand]**

With the preceding choice of $p_\lambda^\mu$, the polynomials $G_\lambda(z, \mathbf{q})$ are Schur positive and give the character of a level two Demazure module of in a highest weight representation of the affine Lie algebra.

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**Theorem[Brito-C-Moura]**

The character of the HL-module associated to a a partition is the same as the (ungraded) character of a level two Demazure module of in a highest weight representation of the affine Lie algebra.

Putting the two together we get:

The characters of the HL-modules are given by $G_\lambda(z,1)$.

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- Why do the Macdonald polynomials show up?

- Why is there a connection between the HL-modules and the Demazure modules ?

In the rest of the talk I want to give some explanation for these things.

# Macdonald polynomials and level one modules.

Let $\mathfrak{g}$ be an arbitrary simple Lie algebra and $\widehat{\mathfrak{g}}$ the corresponding untwisted affine Lie algebra. It has a one-dimensional center spanned by an element $c$ and contains a scaling element $d$ which essentially defines a grading and $\mathfrak{g} \hookrightarrow \widehat{\mathfrak{g}}$.

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Let $\widehat{\mathfrak{b}}$ be a Borel subalgebra and let $\widehat{\mathfrak{p}}$ the standard maximal parabolic subalgebra containing $\widehat{\mathfrak{b}}$. It can be realized as

$$\widehat{\mathfrak{p}} = \mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

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Let $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{b}}$ be the Cartan subalgebra, it can be written as

$$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $W \leq \hat{W}$ be the finite and affine Weyl group.

# Demazure modules

Let $\widehat{P}^+$ be the set of affine dominant integral weights with $\Lambda_0, \cdots, \Lambda_n$ being the affine fundamental weights, here $n$ is the rank of $\mathfrak{g}$,

$$\Lambda_0|\mathfrak{h} = 0 = \Lambda_0(d), \quad \Lambda_0(c) = 1, \quad \Lambda_i = \omega_i + \omega_i(h_\theta)\Lambda_0$$

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Then one knows that $V(\Lambda)$ is a direct sum of eigenspaces with respect to the action of $\widehat{\mathfrak{h}} \subset \widehat{\mathfrak{b}}$; $c$ acts as the scalar $\Lambda(c)$ also known as the level of the representation. The eigenvalues of $d$ are bounded above by $\Lambda(d)$ and the eigenspaces are finite-dimensional.

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For all $w \in \hat{W}$ we have $\dim V(\Lambda)_{w\Lambda} = 1$ and the Demazure module $V_w(\Lambda)$ is the $\widehat{\mathfrak{b}}$-module generated by this weight space. It is easily seen that it is finite-dimensional.

# A result of Sanderson and Ion

Let $\Lambda \in \widehat{P}^+$ and $w \in \widehat{W}$ be chosen so that the restriction of $w\Lambda$ to $\mathfrak{h}$ is in $-P^+$.

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**Theorem[Sanderson, Ion**

Assume that $\mathfrak{g}$ is of type $A, D, E$. Let $w_0 \in W$ be the longest element and $w \in \hat{W}$ be such that $\lambda = -w_0 w \Lambda_0 |\mathfrak{h} \in P^+$. Then the character of the $\mathfrak{g}$-stable Demazure module $V_{w_0 w}(\Lambda_0)$ is $P_\lambda(z, q, 0)$.

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There are two immediate questions which arise from this theorem.

Is there an analogous result for the non-simply laced types?

What can one say about the character of other $\mathfrak{g}$-stable Demazure modules.

# Connection with Quantum Affine Algebras

It was known since the turn of the century when, Sanderson (in type A) and Ion (in type $D, E$) proved their result, that the Macdonald polynomial was too big to be the character of a Demazure module in the non-simply laced cases.

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We called them Weyl modules, but they are also known as standard modules in the literature.

They are just a tensor product of fundamental modules $[\boldsymbol{\omega}_{i_1,q^{r_1}}] \otimes \cdots \otimes [\boldsymbol{\omega}_{i_k,q^{r_k}}]$ taken in a suitable order. Their character only depends on the associated partition $\lambda = \sum_{j=1}^{k} \omega_{i_j}$.

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In the classical ($q \to 1$) limit these modules become reducible, but indecomposable (finite-dimensional) modules for the affine Lie algebra. And they have very nice universal properties.

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**Theorem [CP][C-Loktev],[Fourier-Littelmann]**

If $\mathfrak{g}$ is of type $A, D, E$ the local Weyl module $W_{\rm loc}(\lambda)$ is isomorphic to $V_{w_0w}(\Lambda)$, $\Lambda \in \hat{P}^+$, $\Lambda(c) = 1$ and $w_0w|\mathfrak{h} = -\lambda$. In particular the character of the corresponding standard module for the quantum affine algebra is given by the specialized Macdonald polynomial $P_{-w_0w\lambda}(z, 1, 0)$.

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This theorem shows that the connection with quantum affine alegbras is really crucial.; it was what motivated the definition of the local Weyl modules for the current algebra.

But this theorem does not a tell us the character of the Demazure module is in the non-simply laced case.

# A result of Naoi

To get to the character of the Demazure module, we need the following.

**Theorem[Naoi]**

Suppose that $\mathfrak{g}$ is of type $B, C, F, G$. Then $W_{\mathrm{loc}}(\lambda)$ has a decreasing filtration:

$$W_{\mathrm{loc}}(\lambda) \supset W_1 \supset W_2 \supset \ W_r = \{0\}$$

and

$$W_j / W_{j+1} \cong V_{-w_0 w_j}(\Lambda)$$

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$$W_j/W_{j+1} \cong V_{-w_0 w_j}(\Lambda)$$

for some $\Lambda \in \hat{P}^+$ with $\Lambda(c) = 1$.

Such a filtration is called a level one Demazure flag. The only time $r = 1$ is when $\lambda$ takes values $0, 1$ on all the short simple roots.

# A result of Naoi

It can be proved that $V_{-w_0w}(\Lambda)$ actually depends only on the pair $(-w_0w\Lambda|\mathfrak{h}, \Lambda(c))$. So it is convenient to denote these modules by $D(\ell, \lambda)$, where $-w_0w\Lambda = \lambda$ and $\ell = \Lambda(c)$.

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As a consequence of C-Ion and Naoi's result, we can write

$$P_\lambda(z, \mathbf{q}, 0) = \sum_{\mu \in P} [W_{\text{loc}}(\lambda) : D(1, \mu)]_{\mathbf{q}} \text{ch}_{\text{gr}} D(1, \mu),$$

where

$$[W_{\text{loc}}(\lambda) : D(1, \mu)]_q = 0$$

unless $\mu \leq \lambda$ and $[W_{\text{loc}}(\lambda) : D(1, \lambda)]_q = 1$.

# A reduction to type $A$

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Given $\lambda \in P_{\mathfrak{g}}^+$ let $\lambda_s$ be the restriction of $\lambda$ to the short simple roots, in particular $\lambda_s \in P_{\mathfrak{sl}_r}^+$.

**Theorem[Naoi]**

For $\mathfrak{g}$ of type $B, C, F, G$ we have

$$[W_{\mathrm{loc}}(\lambda) : D(1,\mu)]_{\mathfrak{g}} = \delta_{\lambda-\lambda_s,\mu-\mu_s}[W_{\mathrm{loc}}(\lambda_s) : D(d,\mu_s)]_{\mathfrak{sl}_r},$$

where $d = 3$ if $\mathfrak{g}$ is of type $G_2$ and $d = 2$ otherwise.

# Level two Demazure modules for $A_n$

So, in view of Naoi's result, one really wants to know the polynomials $[W_{\rm loc}(\lambda) : D(2,\mu)]_q$ for $A_n$, in one words using Sanderson, we can write

$$P_\lambda(z, \mathbf{q}, 0) = \mathrm{ch}_{\rm gr} W_{\rm loc}(\lambda) = \sum [W_{\rm loc}(\lambda) : D(2,\mu)]_{\mathbf{q}} \mathrm{ch}_{\rm gr} D(2,\mu).$$

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Compare it with

$$P_\lambda(z,\mathbf{q},0) = \sum_\mu p_\lambda^\mu(\mathbf{q}) G_\mu(z,\mathbf{q}).$$

So the Schur positivity of $G_\mu(z,\mathbf{q})$ follows by showing:

For $\mu \in P^+$ we have

$$G_\mu(z,\mathbf{q}) = \mathrm{ch}_{\rm gr} D(2,\mu), \quad p_\lambda^\mu = [W_{\rm loc}(\lambda) : D(2,\mu)]_{\mathbf{q}}.$$

# The explicit formula for $p_\lambda^\mu$

Recall that we had

$$p_\lambda^\mu(\mathbf{q}) = \mathbf{q}^{\frac{1}{2}(\lambda+\mu_1,\ \lambda-\mu)} \prod_{j=1}^{n} \begin{bmatrix} (\lambda-\mu,\ \omega_j)+(\mu_0,\alpha_j) \\ (\lambda-\mu,\omega_j) \end{bmatrix}_{\mathbf{q}}.$$

Our convention is that $\begin{bmatrix} n \\ m \end{bmatrix}_{\mathbf{q}} =$ if $m < 0$ or $m > n$.

But I have no good reason for this formula! We showed that the existence of such a formula forced $[W_{\mathrm{loc}}(\lambda) : D(2,\mu)]_{\mathbf{q}}$ to satisfy certain recursive relations We guessed the closed formula by using Sage. And eventually succeeded in proving it.

# Back to HL-modules

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The character of the quantum module does not change on passing to the classical limit.

And so: the character of the HL-module associated to a partition $\lambda$ is $G_\lambda(z, 1, 0)$.

Thank you for your attention.