Generic Bases for Surface Cluster Algebras

jt. w/ Jan Schröer (Bonn)

& Daniel Labardini-Fragoso (UNAM-CDMX)

Plan.

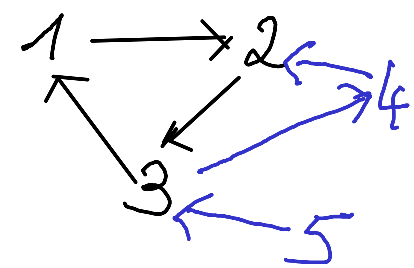
1. Caldero-Chapoton (CC)-functions w coeff.
2. Generically τ-reduced Cp. & generic CC fcn
3. Surface cluster algebras & generic bases
4. Surfaces without punctures and bangles

1. CC-functions with coefficients

$\tilde{Q}$: a acyclic ice quiver
with mutable vertices $\{1,2,\dots,n\} = \underline{n}$
frozen vertices $\{n+1, n+2, \dots, m\}$

$\cup$

Q full subquiver of mutable vert



$\mathbb{C}\langle\langle \tilde{Q} \rangle\rangle$ completed path algebra

$\tilde{W} \in \mathbb{C}\langle\langle \tilde{Q} \rangle\rangle_{cyc}$ non deg. potential

$\tilde{\mathcal{P}} = \mathcal{P}(\tilde{Q}, \tilde{W}) := \mathbb{C}\langle\langle \tilde{Q} \rangle\rangle / \overline{\langle \partial.\tilde{W} \rangle}$ Jacobian alg

$W = \tilde{W}|_Q$ restriction, non-deg!

$\mathcal{P} = \mathcal{P}(Q, W) \twoheadleftarrow \mathcal{P}(\tilde{Q}, \tilde{W})$

$\therefore$ {Rep of $\tilde{\mathcal{P}}$ w support in $\underline{n}$} = {repn of $\mathcal{P}$}

$\mathcal{M} = (M, V)$ decorated repn. of $\mathcal{P}$:

M: rep. of $\mathcal{P}$, V: Q_0-graded VS

$0 \to M \to \tilde{I}_0 \to \tilde{I}_1$ min inj copres as $\tilde{\mathcal{P}}$-rep

$\tilde{g}_{\mathcal{M}} := [\tilde{I}_1] - [\tilde{I}_0] + \underline{\dim} V \in \mathbb{Z}^m = K_0(\tilde{\mathcal{P}}\text{-inj})$

extended g-vector of $\mathcal{M}$

<u>Lemma</u> $\tilde{g}_{\mathcal{M}}|_{\mathbb{Z}^n} = g_{\mathcal{M}}$ & $(\tilde{g}_{\mathcal{M}})_i \geq 0$ for $i > n$

$\tilde{B}_{\tilde{Q}} \in \mathbb{Z}^{m \times n}$ w $(\tilde{B}_{\tilde{Q}})_{ij} = \#\{i \xleftarrow{\tilde{Q}} j\} - \#\{i \xrightarrow{\tilde{Q}} j\}$

$$CC_{\tilde{\mathcal{P}}}(\mathcal{M}) := \underline{x}^{\tilde{g}_{\mathcal{M}}} \sum_{\underline{e} \in \mathbb{N}_0^n} \chi(\mathrm{Gr}_{\underline{e}}^{\mathcal{P}}(M))\, \underline{x}^{\tilde{B}_{\tilde{Q}}\underline{e}} \in \underbrace{\mathbb{Q}[x_1^{\pm 1}, \dots, x_m^{\pm 1}]}_{R}$$

'Caldero-Chapoton fcn w coeff'

$\mathcal{C}(\tilde{Q}, \tilde{W})$ $\mathbb{C}$-span of all $CC_{\tilde{\mathcal{P}}}(\mathcal{M}) \mid \mathcal{M} \in \mathrm{DecRep}(\tilde{\mathcal{P}})$

$\supseteq$

$\mathbb{Q}[x_{n+1}^{\pm}, \dots, x_m^{\pm}]$

<u>Lemma ([DWZ])</u> • $\mathcal{C}(\tilde{Q}, \tilde{W})$ is in fact a $\mathbb{C}$-<u>algebra</u>

$$\mathcal{A}(\tilde{Q}) \subset \mathcal{C}(\tilde{Q}, \tilde{W}) \subset \mathcal{U}(\tilde{Q})$$

<u>Pf.</u> • $CC_{\tilde{\mathcal{P}}}(\mathcal{M} \oplus \mathcal{N}) = CC_{\tilde{\mathcal{P}}}(\mathcal{M}) \cdot CC_{\tilde{\mathcal{P}}}(\mathcal{N})$

• $CC_{\mu_k(\tilde{\mathcal{P}})}(\mu_k(\mathcal{M}))\big|_{x_k \to \mu_k(x_k)} = CC_{\tilde{\mathcal{P}}}(\mathcal{M})$ [Key Lemma]

2. Generically τ-reduced irreducible comp. & generic CC-functions

For $\mathcal{M}, \mathcal{N} \in \mathrm{DecRep}(P)$ one sets ([DWZ])

$E(\mathcal{M}, \mathcal{N}) := \dim \mathrm{Hom}_P(M, N) + (\underline{\dim} M) \cdot \underline{g}_{\mathcal{N}}$ $\quad E(\mathcal{M}) = E(\mathcal{M}, \mathcal{M})$

For $\underline{d}, \underline{v} \in \mathbb{N}_0^n$ we define the affine scheme of decorated representations with dim vector (d, v)

$$\mathrm{DecRep}_{(\underline{d}, \underline{v})}(P) \overset{\text{conj}}{\circlearrowleft} \mathrm{Gl}_{\underline{d}} = \times \mathrm{Gl}_{d_i}(\mathbb{C})$$

An irreducible cp. $Z \in \mathrm{IrrDecRep}_{(\underline{d}, \underline{v})}(P)$ is **generically τ-reduced** if $\exists\, \mathcal{U} \subset Z$ dense s.t.

$$\mathrm{codim}_Z (\mathrm{Gl}_{\underline{d}} \cdot X) = E(X) \quad \forall X \in \mathcal{U}$$

5

$$\mathcal{G}(\tilde{Q},\tilde{W}) := \{ CC^{gen}_{\tilde{\mathcal{P}}}(Z) \mid Z \in \underbrace{\mathrm{IrrDecRep}^{\tau}(\mathcal{P})}_{\bigcup_{(\underline{d},\underline{v}) \in (\mathbb{N}_0^n)^2} \mathrm{IrrDecRep}^{\tau}_{(\underline{d},\underline{v})}(\mathcal{P})} \} \subset \mathcal{C}(\tilde{Q},\tilde{W})$$

<u>Lemma</u>: $\mathcal{G}(\tilde{Q},\tilde{W})$ depends only on the mutation class of $(\tilde{Q},\tilde{W})$ and it contains all cluster monomials.

$\rightsquigarrow$ set of generic CC-functions

$\rightsquigarrow$ generic basis if $\mathbb{C}$-basis of $\mathcal{C}(\tilde{Q},\tilde{W})$

Rem: Generalizes a result of [P] since we allow $\tilde{\mathcal{P}}$ infinite dim.

For proof: If $Z \in \mathrm{IrrDecRep}^{\tau}(\tilde{Q},\tilde{W})$, $k \in \underline{n}$

$\exists! Z' \in \mathrm{IrrDecRep}^{\tau}(\mu_k(\hat{Q},\hat{W}))$ $\quad \mathcal{U} \overset{\text{dense open}}{\subset} Z$, $\mathcal{U}' \overset{\text{dense open}}{\subset} Z'$

$\forall X \in \mathcal{U}: \mu_k(X) \cong X' \in \mathcal{U}' \Longleftrightarrow \quad + \text{Lemma}$

Thm (Plamondon, CLFS)

$g^{gen}: \mathrm{IrrDecRep}(\hat{P}) \longrightarrow \mathbb{Z}^n, \quad Z \mapsto g^{gen}_Z$
is an injective map. It is in fact bijective if P is f.d.

$\square$

Thm 1 Suppose that $\mathrm{Ker}(B_Q) \cap \mathbb{N}_0^n = \{0\}$, (*)
then for each family $(M_i)_{i\in I}$ in $\mathrm{DecRep}(P)\setminus\{0\}$ with pairwise different g-vectors the family $(CC_{\hat{P}}(M_i))_{i\in I}$ in R is $C = \mathbb{Q}[x_{n+1}^{\pm}, \ldots, x_m^{\pm}]$ linear indep.

In part. if $Q' \overset{mut}{\sim} Q$ fulfills (*) then $\mathcal{G}(\tilde{Q}, \tilde{W})$ is a C-lin indep. family.

<u>Corollary</u> Suppose Q fulfills the foll.:

(1) Q is locally acyclic via G. Muller's Banff Alg

(2) $\exists\, Q' \overset{mut}{\sim} Q$ which fulfills $(*)$

(3) $\exists\, Q'' \overset{mut}{\sim} Q$ such that Q'' admits a green to red sequence

$\Longrightarrow$ $\mathcal{G}(\tilde{Q}, \tilde{W})$ is a generic basis of $\mathcal{A}(\tilde{Q})$ $\forall\, \tilde{Q}$ ice quiver with $\tilde{Q}_{mut} = Q$.

Pf • DWZ-Lemma + (1) $\Longrightarrow$ $\mathcal{A}(\tilde{Q}) = \mathcal{C}(\tilde{Q}, \tilde{W}) = \mathcal{U}(\tilde{Q})$

• By (3) + Fan Qin $\mathcal{G}(\tilde{Q}_{prin}, \tilde{W})$ basis of $\mathcal{U}(\tilde{Q}_{prin})$

$\xRightarrow{\text{version of } \mathcal{C}}$ $\mathcal{G}(\tilde{Q}, \tilde{W})$ spans $\mathcal{C}(\tilde{Q}, \tilde{W})$

• (2) $\Longrightarrow$ $\mathcal{G}(\tilde{Q}, \tilde{W})$ is $\mathbb{C}$-lin ind

$\square$

3. Surface Cluster Algebras [FST]

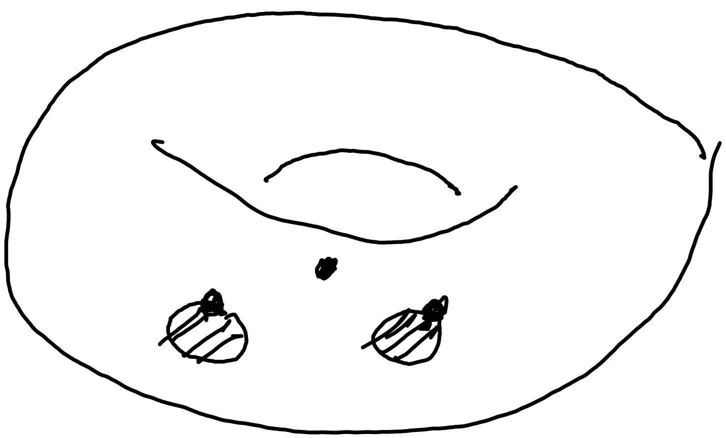
(Σ, M) bordered surface w/ marked pts.

Σ : oriented, connected (Riemann) surface

consider $\partial\Sigma \subset \Sigma$

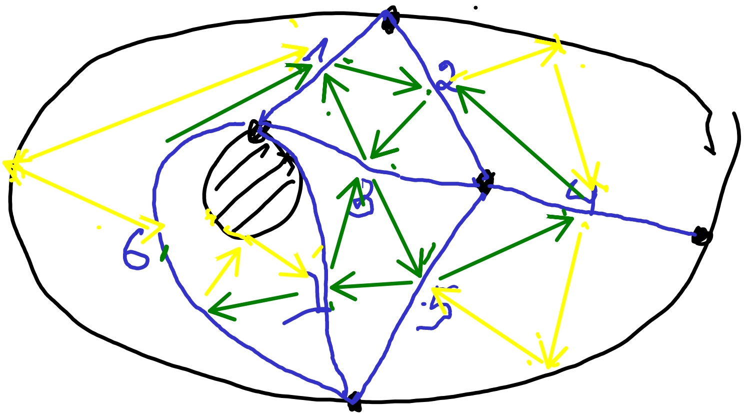
$\cup$

M finite subset $M \cap \sigma \neq \emptyset \;\; \forall \underset{}{\overset{\text{conn. cp.}}{\sigma}} \subset \partial \Sigma$



A triangulation $T = (\tau_1, \tau_2, \dots, \tau_n)$ of (Σ, M) is a maximal collection of (tagged) arcs connecting the pts of M without intersecting each other and itself (bdry segs are not arcs)

Triangulation $T \rightsquigarrow$ quiver Q_T: vertices — arcs
int Δ — clockwise oriented 3-cycles



$\tilde{Q}_T$ ice quiver
(any geom coeff)

core $B_T = |\{\text{punct.}\}| + |\{\text{bdry. cp. with even \# marked pts}\}|$

- Flip of diag $\boxed{\diagdown} \leftrightarrow \boxed{\diagup}$ — FZ mutation of Q_T
- different Δs yield mutation equiv quivers
- most mutation finite quivers occur that way

Interested in $\mathcal{A}(\tilde{Q}_T)$ surface cluster algs

<u>Obs</u> $\operatorname{Spec}(\mathcal{A}(\tilde{Q}_T))(\mathbb{R}_{>0}) \simeq$ Decorated laminated Teichmüller space of (Σ, M).

Particularly nice situation if $|\mathbb{M} \cap \partial\Sigma| \geq 2$:

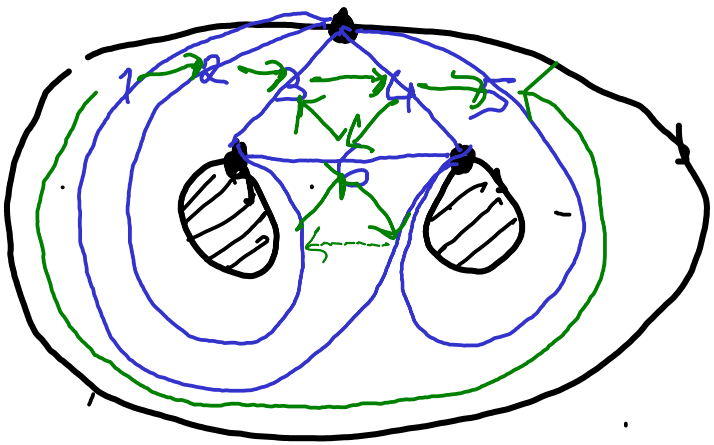
- Q_T is "Banff" locally acyclic (G. Muller)
- Q_T admits a "unique" non deg potential (GLFS)
- Q_T admits a max green sequence [---]

$\ast$ Can find triang T s.t. Q_T fulfills $(\ast)$

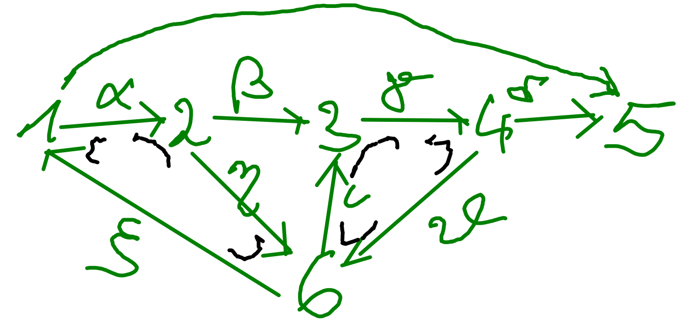
Cor $\Longrightarrow$ $\mathcal{A}(\tilde{Q}_T)$ has a (unique) generic basis

Rest of talk: Compare $\mathcal{G}(\tilde{Q}_T, \tilde{W}_T)$ with MSW bangle basis for $\mathbb{M} \subset \partial\Sigma$

4. Surface cluster algebras w/o punctures



Q_T :

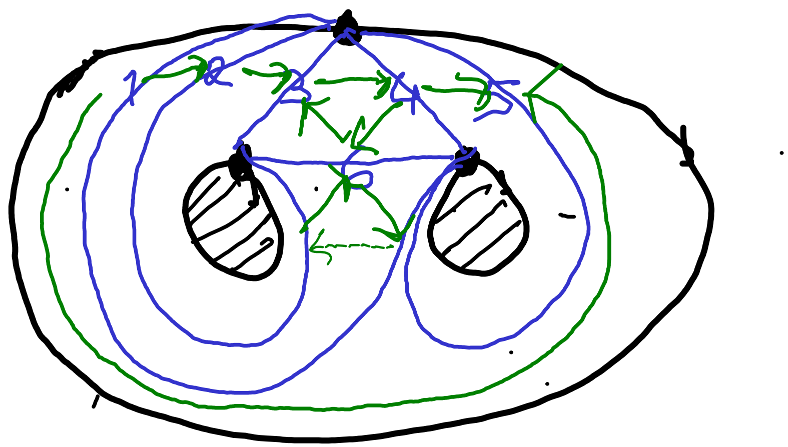


$$W_T = \xi\eta\alpha + \varepsilon\vartheta\gamma$$

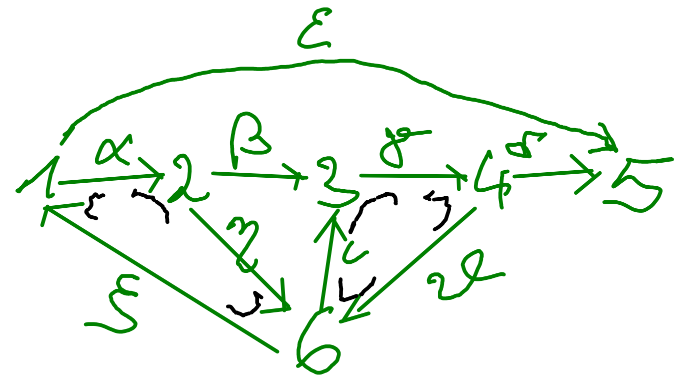
<u>Thm [B___]</u> If (Σ, M) has no punctures, $\mathcal{P}(Q_T, W_T)$ is a gentle algebra for each triang T.

⇝ classification of indec repn. in terms of bands and strings

⟷ homotopy classes of (closed) curves



Q_τ :



$$W_\tau = \xi\eta\alpha + \iota\vartheta\gamma$$

Also the description of the (affine) scheme

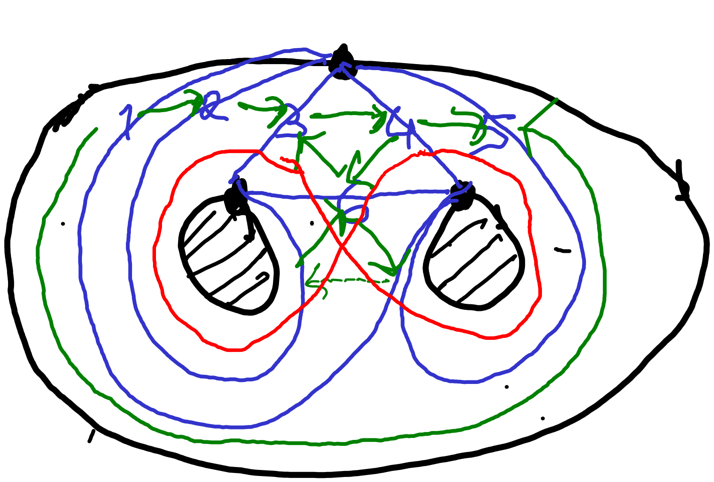
$\mathrm{Rep}_{\underline{d}}(\mathcal{P}) \overset{\text{conj}}{\circlearrowleft} GL_{\underline{d}} = \prod_{i=1}^{n} GL_{d_i}$ is easy. In our ex:

$$\mathrm{Rep}_{\underline{d}}(\mathcal{P}) \cong \mathrm{Rep}_{d_1 d_2 d_6}\left(\begin{tikzcd}[sep=small] 1 \arrow[r] & 2 \arrow[dl] \\ 6 \arrow[u] & \end{tikzcd}\right) \times \mathrm{Rep}_{d_2 d_3}(2 \xrightarrow{\beta} 3) \times \mathrm{Rep}_{d_3 d_4 d_6}\left(\begin{tikzcd}[sep=small] 3 \arrow[r] & 4 \arrow[dl] \\ 6 \arrow[u] & \end{tikzcd}\right)$$
$$\times \mathrm{Rep}_{d_1 d_5}(1 \xrightarrow{\delta} 5) \times \mathrm{Rep}_{d_4 d_5}(4 \xrightarrow{\varepsilon} 5)$$

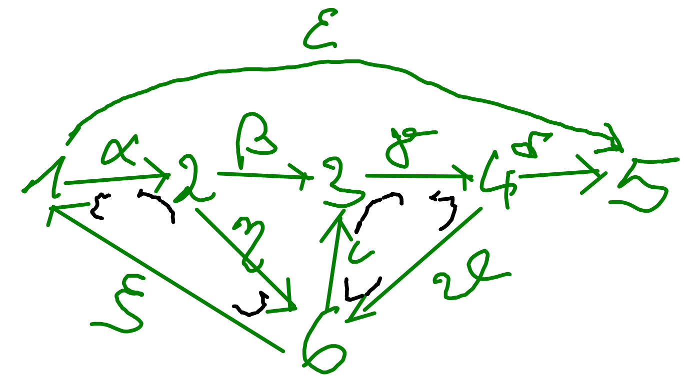
<u>Thm</u> $Z \in \mathrm{IrrDecRep}(\mathcal{P})$ is generically τ-reduced

$\Longleftrightarrow$ $\forall$ proj to "W-blocks" is generically τ red. (!)

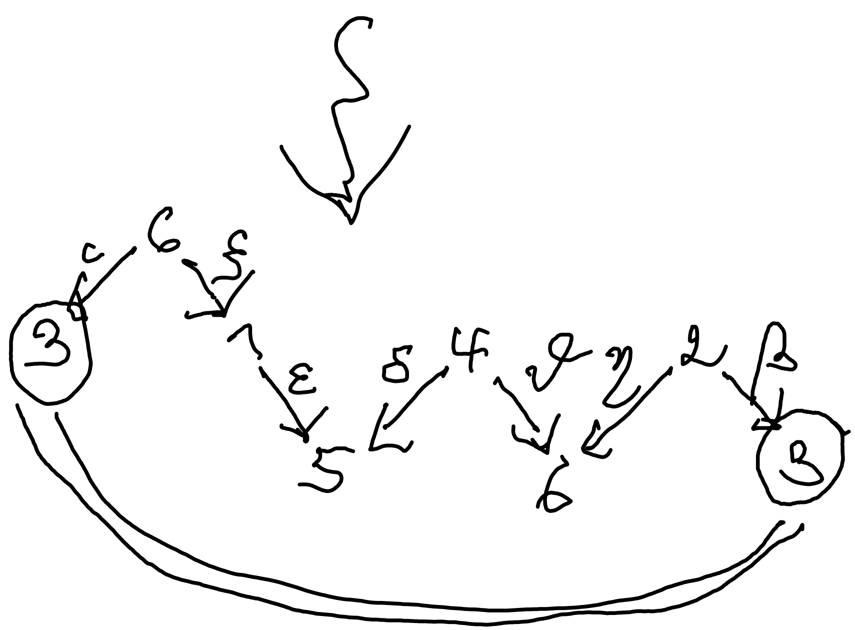
Example:



Q_T :



$$W_T = \xi\eta\alpha + \iota\vartheta\gamma$$



Band module

(24)

Want a more concrete description of $\mathcal{G}(\hat{\tilde{Q}}_T, \tilde{W}_T)$

$Z \in \mathrm{Irr}^{\tau}\mathrm{DecRep}(\mathcal{J})$ is indecomposable if it contains a dense open subset of indec repns

[CBS] [CLFS]

Each irred τ-reduced$^{(gen)}$ component is "**direct** sum" of indec irred τ-red. cps.

$Z = \overline{Z_1 \oplus Z_2} \iff E^{gen}(Z_1, Z_2) = 0$

<u>Theorem [BZ], [GLFS]</u> for (Σ, M) w/o punctures

a) The indecomposable gen τ-red cp. with an open orbit correspond$^{1:1}$ to arcs on (Σ, M)

b) The remaining indec gen. τ-red. cps. contain a 1-param. family of bricks and are parametrized by simple loops

c) Two such indec. cps are E-orthogonal (i.e. can be summed up) iff the corresp. curves do not intersect (nontrivially),

$\therefore \ \mathrm{Irr}^{\tau}\mathrm{DecRep}(\mathcal{P}) \overset{1:1}{\longleftrightarrow} \mathrm{Lam}(\Sigma, M)$

snake / easy graphs

$$
\begin{tikzcd}
\{\text{cluster monomials in } \mathcal{A}(\Sigma, M)\} \arrow[r, phantom, "\subseteq"] \arrow[d, leftrightarrow, "1:1"', "{[FST]}"] & \mathcal{MSW}_T \arrow[d, leftrightarrow, "1:1"', "\text{def}"] & \\
\{L \in \mathrm{Lam}(\Sigma, M) \mid L \text{ has no loops}\} \arrow[r, phantom, "\subseteq"] \arrow[d, leftrightarrow, "1:1"', "{[BZ], [AIR]}"] & \mathrm{Lam}(\Sigma, M) \arrow[r, leftrightarrow, "1:1", "\text{shear}"'] & \mathbb{Z}^n \arrow[d, equal] \\
\{Z \in \mathrm{InvDecPr}^{\tau}(\mathcal{P}(Q_T, W_T)) \mid Z \text{ dense orbit}\} \arrow[r, phantom, "\subseteq"] \arrow[d, leftrightarrow, "1:1"', "{[DWZ]}"] & \mathrm{InvDecPr}^{\tau}(\mathcal{P}) \arrow[u, leftrightarrow, dashed, "{[G, LFS]}"'] \arrow[r, leftrightarrow, "g\text{-vec}"] \arrow[d, leftrightarrow, "1:1"', "\text{Def}"] & \mathbb{Z}^n \\
\{\text{cluster monomials in } \mathcal{A}(Q_T)\} \arrow[r, phantom, "\subseteq"] & \mathcal{G}(Q_T, W_T) &
\end{tikzcd}
$$

[MSW]

Thurston

Bases

[Haupt]