Support of the spherical module of the rational Cherednik algebra

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Journées d'algèbre
À l'occasion des 60 ans de Bernard Leclerc
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Outline

1. Complex reflections groups, braid groups, Hecke algebras

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2. Rational Cherednik algebras

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3. Etingof's criterion for Coxeter groups

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Conjugacy classes of reflections: $r_{H,j} := r_H^j$, $H \in [\mathcal{A}/W]$, $1 \leq j \leq n_H - 1$.

Classification of complex reflection groups

Theorem (Shephard-Todd)

If (W, V) is an irreducible complex reflection group, then either:

- $W \simeq G(de, e, n)$, the group of $n \times n$ monomial matrices whose non-zero entries lie in μ_{de} and have product in μ_d;

  (particular case $G(1, 1, n) = \mathfrak{S}_n$ has rank $n - 1$)

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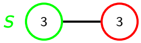
$G_4 =$ 

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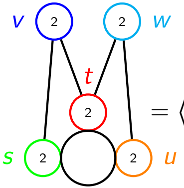
$G_{31} =$ 

$$= \left\langle \begin{array}{c} s,\ t,\ u,\ v,\ w \mid \\ sut = uts = tsu, \\ uv = vu, sw = ws, vw = wv, \\ svs = vsv, tvt = vtv, twt = wtw, wuw = uwu, \\ s^2 = t^2 = u^2 = v^2 = w^2 = 1 \end{array} \right\rangle$$

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B_W has an "Artin-like" presentation such that one obtains a presentation of W by adding the order relations (Broué-Malle-Rouquier, Bessis-Michel, Bessis).

Hecke algebras of complex reflection groups

To get the Hecke algebra $\mathcal{H}$ of the complex reflection group W from the group algebra of its braid group, we deform the order relations:

$$(\sigma_H - u_{H,0}) \dots (\sigma_H - u_{H,n_H-1}) = 0$$

over the ring $\mathbb{Z}[\mathbf{u}^{\pm 1}] = \mathbb{Z}[u_{H,j}^{\pm 1}]$, for $H \in [\mathcal{A}/W]$.

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1. The algebra $\mathcal{H}$ is a free $\mathbb{Z}[\mathbf{u}^{\pm 1}]$-module of rank $|W|$.
2. Moreover, there is a linear form $t : \mathcal{H} \longrightarrow \mathbb{Z}[\mathbf{u}^{\pm 1}]$ such that:
   - t is a symmetrizing form on $\mathcal{H}$, i.e., $t(hh') = t(h'h)$ for all $h, h' \in \mathcal{H}$, and $\hat{t} : \mathcal{H} \xrightarrow{\sim} \operatorname{Hom}(\mathcal{H}, \mathbb{Z}[\mathbf{u}^{\pm 1}])$, $h \mapsto (h' \mapsto t(hh'))$.

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   - $\mathcal{H}$ specializes via $u_{H,j} \mapsto e^{2\pi i j/n_H}$ to the canonical symmetrizing form (coefficient of 1) on the group algebra of W.
   - For $(\alpha \mapsto \alpha^*)$ = simultaneous inversion of the indeterminates,

$$\forall b \in B, \quad t(b^{-1})^* = \frac{t(b\tau)}{t(\tau)}, \quad \text{where } \tau = (t \mapsto v_0 e^{2\pi i t}) \in Z(P).$$

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The **rational Cherednik algebra** $H_c(W, V)$ is the subalgebra of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[V])$ generated by the group algebra $\mathbb{C}W$, the polynomial algebra $\mathbb{C}[V]$ acting by multiplication, and the Dunkl operators

$$y(f) = \partial_y(f) - \sum_{r \in R} c_r \langle \alpha_r, y \rangle \frac{f - r(f)}{\alpha_r}, \text{ for } y \in V \text{ and } f \in \mathbb{C}[V].$$

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We may write $c = (c_{H,j})$.

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Better: $c_{H,\chi} := n_H^{-1} \sum_{r \in W_H} c_r(1 - \chi(r))$, $H \in [\mathcal{A}/W]$, $\chi \in \widehat{W_H} \setminus \{\mathbf{1}\}$.

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Theorem (Etingof-Ginzburg: PBW for symplectic reflection algebras)

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Category $\mathcal{O}_c = \mathcal{O}_c(W, V)$ is the full subcategory of $H_c(W, V)$–mod consisting of modules that are

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The standard module corresponding to $E \in \operatorname{Irr} \mathbb{C}W$ is

$$\Delta_c(E) := H_c(W, V) \otimes_{\mathbb{C}[V^*] \rtimes \mathbb{C}W} E \qquad (\simeq \mathbb{C}[V] \otimes_{\mathbb{C}} E \text{ over } \mathbb{C}W \ltimes \mathbb{C}[V]),$$

where Dunkl operators V act by 0 on E.

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$$\operatorname{Rad} \Delta_c(E) \in \mathcal{O}_c^{> c_E} := \{M \in \mathcal{O}_c \mid M^d \neq 0 \Rightarrow d \in c_E + \mathbb{Z}_{>0}\}$$

Highest weight category and c-order

Theorem [Ginzburg-Guay-Opdam-Rouquier]

$\mathcal{O}_c$ is a highest weight category on the poset $(\operatorname{Irr} \mathbb{C}W, \leq_c)$, where

$$E >_c F \Longleftrightarrow c_E - c_F \in \mathbb{Z}_{>0},$$

with standard objects $\Delta_c(E)$.

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Also, each $L_c(E)$ has a projective cover $P_c(E)$ which is Δ-filtered:

$$\begin{cases} 0 \to K_c(E) \to P_c(E) \to \Delta_c(E) \to 0, \\ (K_c(E) : \Delta_c(F)) \neq 0 \Longrightarrow F <_c E. \end{cases}$$

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Definition

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Theorem (Varagnolo–Vasserot)

For W Weyl group and c constant, $\dim L_c(\mathbf{1}) < \infty \iff c \in \mathbb{Q}_{>0}$ with denominator a regular elliptic number for W.

Outline

Coxeter groups, case of equal parameters

Assume (W, S) is a Coxeter system.

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i.e., iff $c \in \mathbb{Q}_{>0}$ with denominator exactly m, such that

$$\#\{i \mid m \text{ divides } d_i(W)\} > \#\{i \mid m \text{ divides } d_i(W')\}$$

Coxeter groups, case of equal parameters

Example: F_4, equal parameters

$$P_W = [2][6][8][12] = \Phi_2^4\Phi_3^2\Phi_4^2\Phi_6^2\Phi_8\Phi_{12}$$

W'	$P_{W'}$	$\lvert W : W'\rvert_q$
B_3, C_3	$[2][4][6] \quad = \Phi_2^3\Phi_3\Phi_4\Phi_6$	$\Phi_2\Phi_3\Phi_4\Phi_6\Phi_8\Phi_{12}$
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```
gap> H:=Hecke(CoxeterGroup("F",4), [x]);
Hecke(F4,x)
gap> SphericalCriterion(H);
[ P2P3P4P6P8P12(x), P2^2P3P4^2P6^2P8P12(x),
P2^2P3P4^2P6^2P8P12(x), P2P3P4P6P8P12(x) ]
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$$P_W = \Phi_2\Phi_3(q_1)\Phi_2\Phi_3(q_2)\Phi_2^2\Phi_4\Phi_6(q_1q_2)\Phi_2(q_1q_2^2)\Phi_2(q_1^2q_2)$$

W'	$\lvert W : W' \rvert_q$
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```
gap> H:=Hecke(F4,[x,x,y,y]);
Hecke(F4,[x,x,y,y])
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$$\begin{aligned}
c_2 &= m, & m &\in \left\{\tfrac{1}{3}, \tfrac{2}{3}\right\} + \mathbb{Z}_{\geq 0}\\
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Common lines: $c_1 + c_2 = \{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}\} + \mathbb{Z}_{\geq 0}$

Intersection points: $\begin{cases} c_2 = \{\frac{1}{3}, \frac{2}{3}\} + \mathbb{Z}_{\geq 0} \\ c_1 + 2c_2 = \frac{1}{2} + \mathbb{Z}_{\geq 0} \end{cases} \cap \begin{cases} c_1 = \{\frac{1}{3}, \frac{2}{3}\} + \mathbb{Z}_{\geq 0} \\ 2c_1 + c_2 = \frac{1}{2} + \mathbb{Z}_{\geq 0} \end{cases}$

Picture for F_4: type B_3 parabolic

$\operatorname{supp} L_c(\mathbf{1}) \not\supseteq V^{W(B_3)}$

$c_2 = m,$

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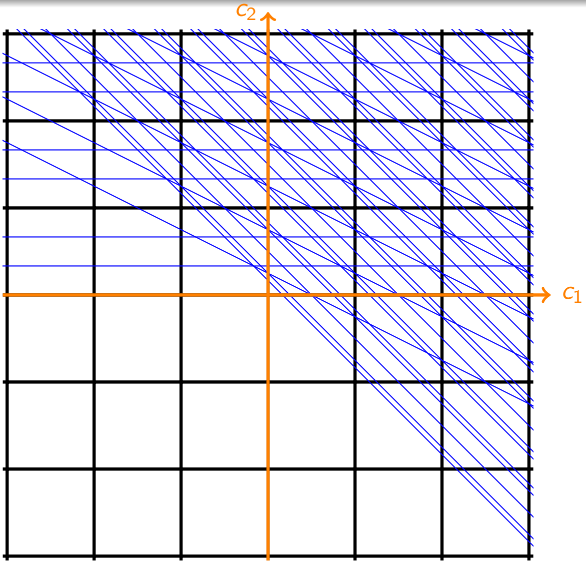

Picture for F_4: type C_3 parabolic

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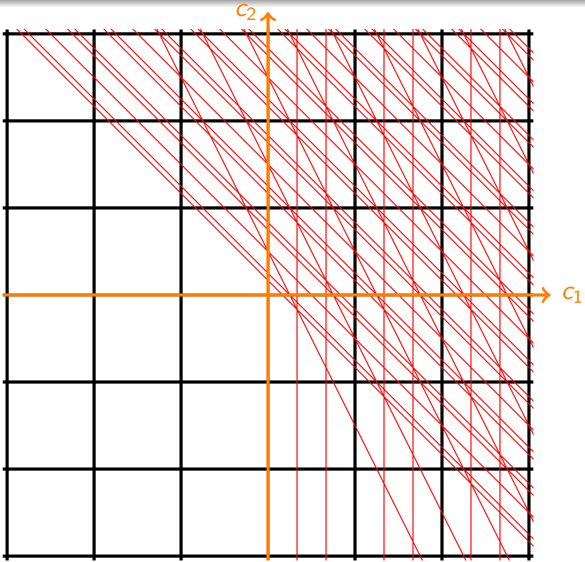

Picture for F_4: intersecting conditions

$\operatorname{supp} L_c(\mathbf{1}) \not\supseteq V^{W(B_3)}$

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common lines

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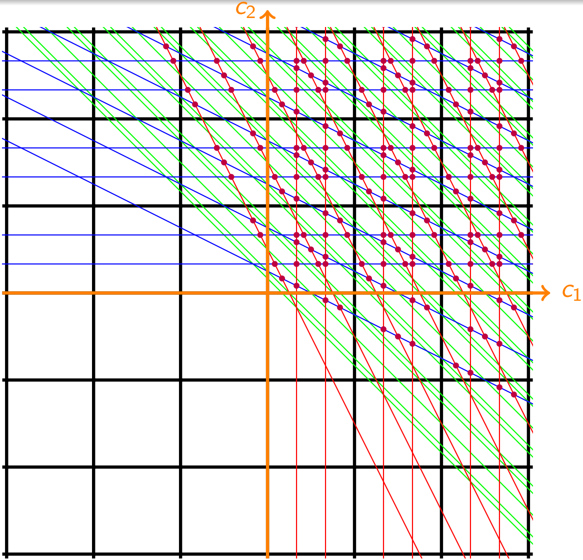
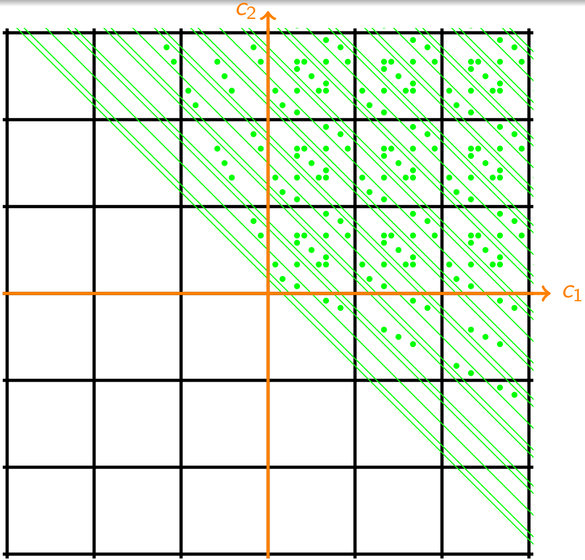
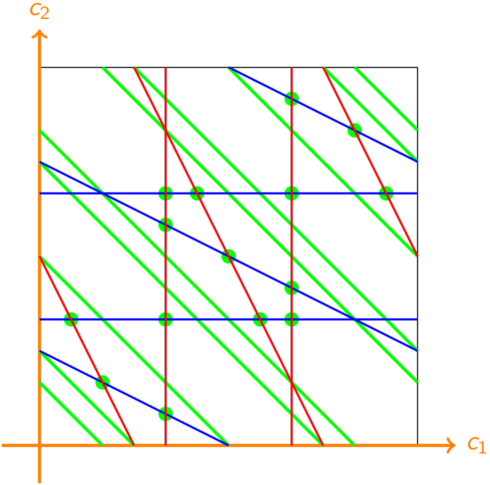


Picture for F_4: finite dimensional locus

$\operatorname{supp} L_c(\mathbf{1}) = \{0\}$



Picture for $F4$: Zooming in the positive part



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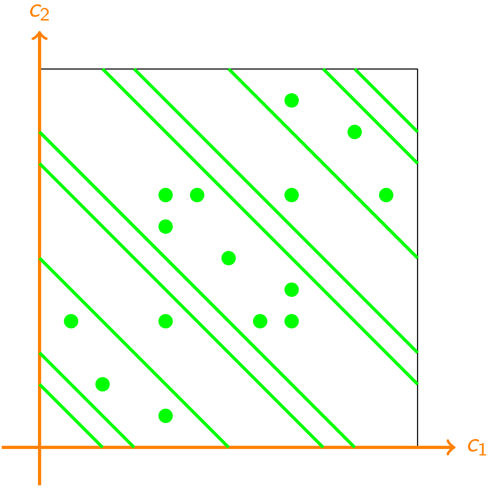

Outline

Functor KZ

Localized to V^{reg}, the rational Cherednik algebra becomes isomorphic to $\mathcal{D}_{V^{\mathrm{reg}}} \rtimes W$. By monodromy, we get the Knizhnik-Zamolochikov functor

$$\mathsf{KZ} : \mathcal{O}_c(W, V) \longrightarrow \mathbb{C}B_W\text{-mod}$$

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The functor KZ factors through an exact functor

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It sends a simple object to a simple object or 0.

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Induction and restriction functors

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Bezrukavnikov and Etingof have defined exact functors

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with adjunctions $(\mathrm{Res}_v, \mathrm{Ind}_v)$ and $(\mathrm{ind}_\lambda, \mathrm{res}_\lambda)$.

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Theorem (Bezrukavnikov-Etingof, Shan, Losev)

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Assuming symmetry of Hecke algebras (with canonical trace):
quotient of principal Schur elements, already computed, and in GAP3!

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Let $P \twoheadrightarrow \Delta \twoheadrightarrow L \in \operatorname{Irr} \mathcal{O}_c$ and $v \in V$.
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(f) $\mathrm{KZ}(P)$ is a direct summand of $\operatorname{Ind}_{\mathcal{H}'_q}^{\mathcal{H}_q} \operatorname{Res}_{\mathcal{H}'_q}^{\mathcal{H}_q} KZ(P)$.

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Moreover, $\mathrm{KZ}\,\Delta_c(\mathbf{1}) = \mathbf{1}_q$. Hence

$$\text{in the region } c \geq 0, \qquad v \notin \operatorname{supp} L_c(\mathbf{1}) \iff \mathbf{1}_q \overset{\oplus}{\not\subset} \operatorname{Ind}_{\mathcal{H}'_q}^{\mathcal{H}_q} \operatorname{Res}_{\mathcal{H}'_q}^{\mathcal{H}_q} \mathbf{1}_q \iff |W : W'|_q = 0$$

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$$v \notin \operatorname{supp} L_c(\mathbf{1}) \Longleftrightarrow \left(\begin{array}{c} c \text{ belongs to a positive hyperplane where} \\ |W : W'|_q \equiv 0 \end{array} \right)$$

Thank you for your attention!

Happy birthday Bernard!

The infinite family $G(r, 1, n)$

Generators of monodromy: $T_0, T_1, \ldots, T_{n-1}$
Parameters: $Q_0, Q_1, \ldots, Q_{r-1}, q$
Hecke relations:

$$\prod_{0 \leq j \leq r-1} (T_0 - Q_j) = 0 \quad \text{and} \quad (T_i + 1)(T_i - q) = 0 \quad \text{for } 1 \leq i \leq n-1$$

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$$\text{Chlouveraki–Jacon: } |G(r, 1, n)|_q = [n]! \prod_{j=1}^{r-1} \prod_{m=0}^{n-1} (q^m Q_0 Q_j^{-1} - 1)$$

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$$|G(r, 1, n) : \mathfrak{S}_k \times G(r, 1, n-k)|_q = \begin{bmatrix} n \\ k \end{bmatrix} \prod_{j=1}^{r-1} \prod_{m=n-k}^{n-1} (q^m Q_0 Q_j^{-1} - 1)$$

The infinite family $G(r, 1, n)$

For $c = (c_0, d_1, \ldots, d_{r-1})$, $q = e^{-2\pi i c_0}$ and $Q_j = e^{2\pi i (j - d_j)/r}$,
$L_c(\mathbf{1})$ is finite dimensional if and only if either

(a) $\exists 1 \leq j \leq r-1$ and $k > 0$ with $k \equiv -j \bmod r$ and

$$d_0 - d_j + r(n-1)c_0 = k,$$

or

(b) $c_0 = \ell/d$ for some $0 \neq d \mid n$ and some $\ell > 0$ coprime to d, and also

$$d_0 - d_j + rmc_0 = k$$

for some $n - d \leq m \leq n-1$ and $1 \leq j \leq r-1$, and and $k > 0$ with $k \equiv -j \bmod r$.

Codimension 2!
cf Gerber–Norton (based on Shan, Shan–Vasserot, Losev)

Exceptional complex reflection groups

We give $|W|_q$ and $|W : W'|_q$ for all maximal parabolic subgroups W' (up to a unit).

Note: some cyclotomic polynomials become reducible over the field of definition.

$\lvert G_4\rvert_q$	$\Phi_2\Phi_3''\Phi_6''(x_1)\Phi_2\Phi_3'\Phi_6'(x_2)\Phi_2(x_1x_2)$
$\lvert G_4 : Z_3\rvert_q$	$\Phi_2\Phi_6''(x_1)\Phi_2\Phi_6'(x_2)\Phi_2(x_1x_2)$
$\lvert G_5\rvert_q$	$\Phi_3''(x_1)\Phi_3'(x_2)\Phi_3''(y_1)\Phi_3'(y_2)\Phi_6''(x_1y_1)\Phi_2(x_1y_2)\Phi_2(x_2y_1)\Phi_6'(x_2y_2)\Phi_2(x_1x_2y_1y_2)$
$\lvert G_5 : Z_3'\rvert_q$	$\Phi_3''(y_1)\Phi_3'(y_2)\Phi_6''(x_1y_1)\Phi_2(x_1y_2)\Phi_2(x_2y_1)\Phi_6'(x_2y_2)\Phi_2(x_1x_2y_1y_2)$
$\lvert G_5 : Z_3''\rvert_q$	$\Phi_3''(x_1)\Phi_3'(x_2)\Phi_6''(x_1y_1)\Phi_2(x_1y_2)\Phi_2(x_2y_1)\Phi_6'(x_2y_2)\Phi_2(x_1x_2y_1y_2)$
$\lvert G_6\rvert_q$	$\Phi_2(x_1)\Phi_3''(y_1)\Phi_3'(y_2)\Phi_2\Phi_6''(x_1y_1)\Phi_2\Phi_6'(x_1y_2)\Phi_2(x_1y_1y_2)$
$\lvert G_6 : A_1\rvert_q$	$\Phi_3''(y_1)\Phi_3'(y_2)\Phi_2\Phi_6''(x_1y_1)\Phi_2\Phi_6'(x_1y_2)\Phi_2(x_1y_1y_2)$
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$\lvert G_7\rvert_q$	$\Phi_2(x_1)\Phi_3''(y_1)\Phi_3'(y_2)\Phi_3''(z_1)\Phi_3'(z_2)\Phi_6''(x_1y_1z_1)\Phi_2(x_1y_1z_2)\Phi_2(x_1y_2z_1)\Phi_6'(x_1y_2z_2)\Phi_2(x_1y_1y_2z_1z_2)$
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etc. . . We thank Gunter Malle, Maria Chlouveraki and Jean Michel!