25 years of LLT polynomials

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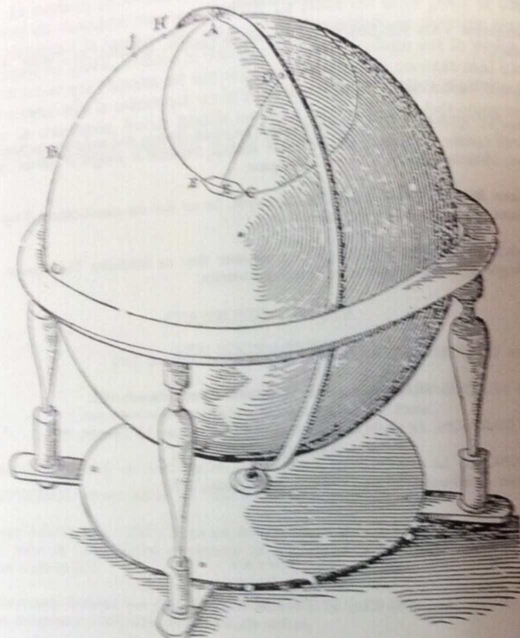
LeclercFest

Préhistoire

Clichy 1987



Hors-texte 1.

Planche du *Quatrieme Livre de la Geographie: de l'Histiodromie, ou cours des Navires*, in Œuvres *Mathématiques* de Simon Stevin de Bruges, édition française d'Albert Girard, Leyde, 1634, p. 143.

HISTOIRE DE LA LOXODROMIE AU XVIIème SIECLE.

Bernard LECLERC
14 juin 1991[0]

"La ligne loxodromique tire son origine, de l'instrument ordinaire, dont on se sert pour la conduite du Vaisseau, c'est-à-dire de la Boussole. Je dis donc que si nous suivons le même Rumb oblique, par exemple le Sudest, la ligne que nous décrirons pendant ce temps, fait des angles de quarante-cinq degrez, non seulement avec le Meridien duquel nous sommes partis; mais encore avec tous ceux que nous rencontrons en chemin, estant une proprieté de l'Aiguille aymantée, de s'ajuster au Meridien du lieu où elle est. On demande donc quelle ligne, fait un angle de quarante-cinq degrez avec tous les Meridiens: & ce sera celle que nous cherchons."

De Chales, *L'Art de Naviger*, 1677.

1. P. Nunes

Pedro Nunes remarqua le premier, dans son *Tratado em defensam da carta de marear* paru en 1537, que la ligne parcourue par un navire qui garde un cap constant n'est pas un arc de grand cercle de la sphère. Claude Millet De Chales nous raconte dans son avant-propos[1] les circonstances de cette découverte:

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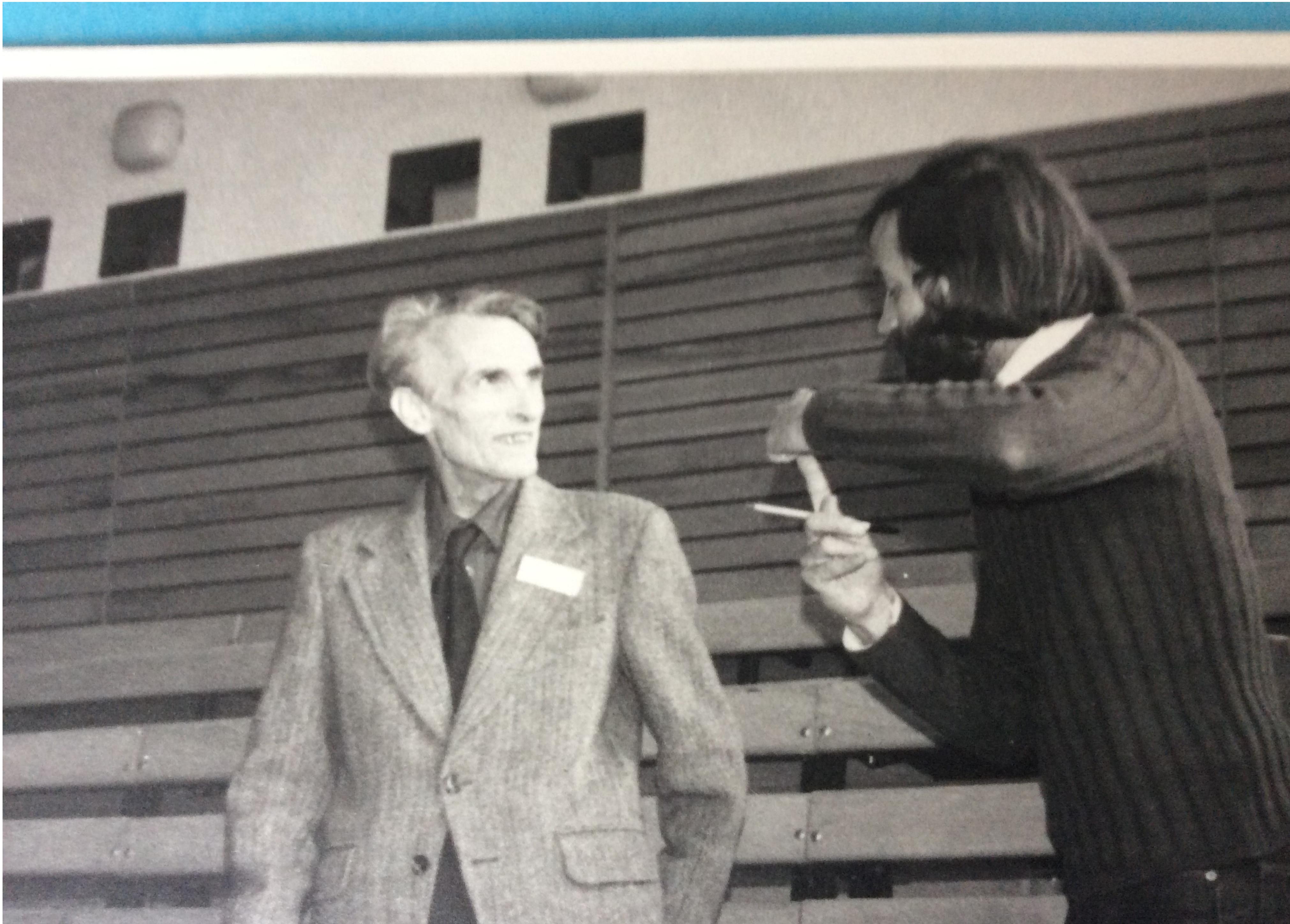
0 Cet article est le texte d'une conférence prononcée le vendredi 14 juin 1991, dans le cadre d'une année du Séminaire Interdisciplinaire d'Histoire des Sciences du Lycée Malherbe, dont le thème était: *Maîtres et Possesseurs de la Nature? Science et techniques aux XVIIème et XVIIIème siècles.*

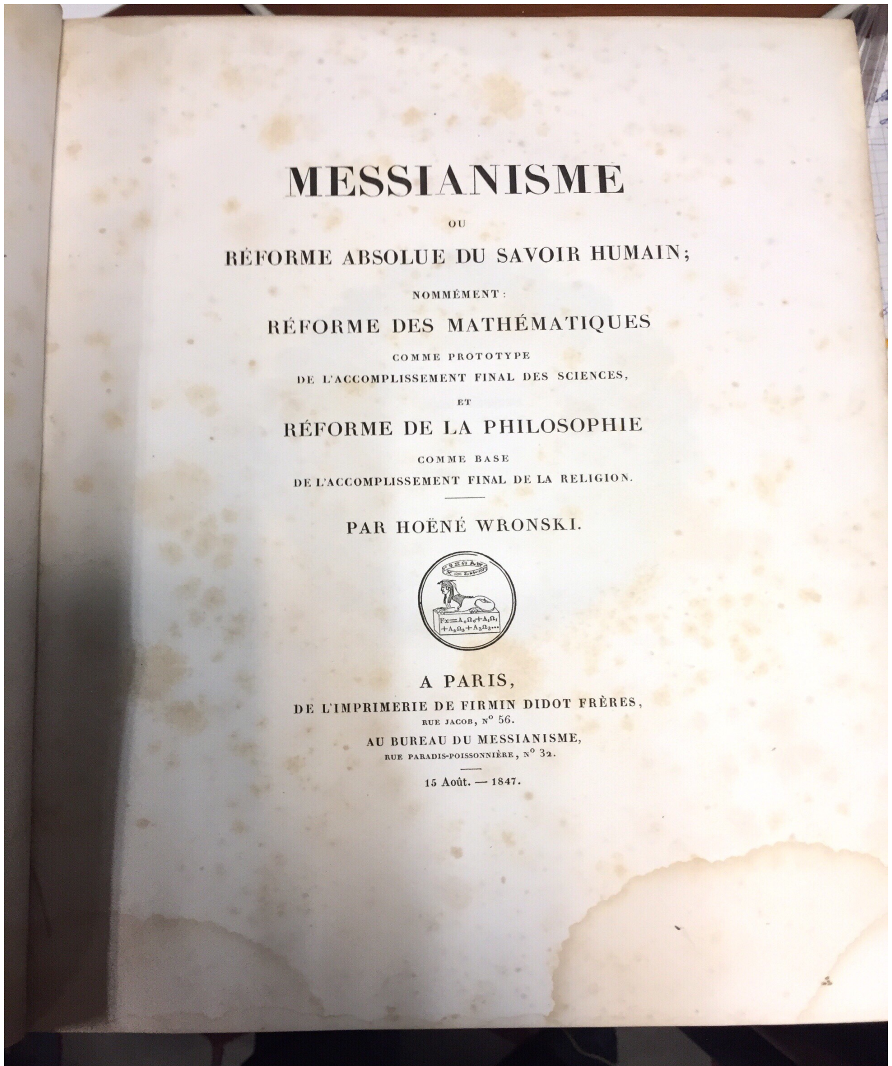
Une première version de ce travail a été rédigée en collaboration avec Marie LETERRIER dans le cadre d'un cours d'Histoire des Sciences donné par Jean-Luc VERLEY à l'Université Paris-VII, en 1990. Cet article leur est dédié.

Certains extraits, en général traduits du latin par l'auteur, ont été adaptés par ses soins, pour le plaisir de façonner (et faire sonner) un verbe français contemporain des textes.

1 *L'Art de naviger démontré par principes & confirmé par plusieurs observations tirées de l'expérience*, Paris, 1677.



Butterfly effect



Cette Résolution des Équations offre, pour l'actuelle Réforme absolue du Savoir humain, tout à la fois, et sa couronne et sa garantie, par la solution philosophique du plus grand problème des sciences, de ce problème mystérieux des Équations que les efforts de tous les savants n'ont pu, jusqu'à ce jour, ni résoudre, ni même comprendre.

RÉSOLUTION GÉNÉRALE

DES

ÉQUATIONS ALGÉBRIQUES

DE TOUS LES DEGRÉS;

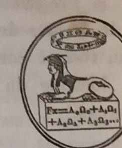
PRÉCÉDÉE DU

MANIFESTE HISTORIQUE

CONCERNANT L'ACTUELLE

RÉFORME DU SAVOIR HUMAIN.

PAR HOËNÉ WRONSKI.



A PARIS,

DE L'IMPRIMERIE DE FIRMIN DIDOT FRÈRES,
RUE JACOB, N° 56.

AU BUREAU DU MESSIANISME,
RUE PARADIS-POISSONNIÈRE, N° 32.

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15 Août. — 1847.

(A).

TOPICS IN ALGEBRA
BANACH CENTER PUBLICATIONS, VOLUME 26, PART 2
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1990

WROŃSKI'S FACTORIZATION OF POLYNOMIALS

ALAIN LASCOUX*

L.I.T.P., Université Paris VIII, Paris, France

In the middle of Wroński's papers [8], [9], [10], [11] one can come across solutions of important problems such as finding, for any polynomial with complex coefficients, the factor which corresponds to roots of modulus less than 1.

1. Background

Józef Maria Hoene-Wroński (1778–1853) was a universal philosopher and scientist. He also knew all languages of culture, Polish, French, Latin, Greek, Hebraic, Arabic, Aramaic, though not English.

His aim was a complete "Réforme du savoir Humain" including both the theory of spontaneous locomotion and the art of governing.

However, his industrial speculations were not bought by the government, nor was his mathematical work accepted by the Academy.

He was therefore compelled to extract (painfully, having even to go to court) money from a banker to publish his philosophical theories. Unfortunately, the finiteness of the banker's fortune and the malevolence of the banker's wife led to a delay of more than 30 years in the publication of his work, apart from a small "Canon des Logarithmes".

Wroński summarizes his object in his "Prolégomènes du Messianisme":

> *"L'objet de cet ouvrage est de fonder péremptoirement la vérité sur la terre, de réaliser ainsi la philosophie absolue, d'accomplir la religion, de réformer les sciences, d'expliquer l'histoire, de découvrir le but suprême des Etats, de fixer les fins absolues ,de l'homme et de dévoiler les destinées des nations"* [10, p. 10].

In fact, according to his own terms,

> *It was with much grief that Hoene-Wroński was forced to leave his grave philosophical tasks to indulge in the Réforme des Mathématiques ... Mathematical questions, however difficult, are only a secondary object, a sort of hobby in the middle of his high philosophical thoughts.* [10, p. 25].

* CNRS, PRC Math-Informatique, Programme Culturel PROCOPE.
This paper is in final form and no version of it will be submitted for publication elsewhere.

He was forced to offer his mathematical laws as a proof of the absolute truth of his philosophical and religious Messianic doctrine (that he borrowed from Towiański), the main weakness of philosophy being that Popper's criterion of *falsification* cannot be applied to its statements, leaving place for much *illusions* which Wroński proposes to dispel.

Though restricting his field of research, he was nevertheless able to attain the *Supreme Law of mathematics, which contains all known mathematics as a very special case and extends indefinitely beyond what is known.* Moreover, his *law would also comprise all formulas and all methods which can be obtained in the future of Science* [10, p. 32].

Wroński almost instantly met with the outright hostility of the "*savants sur brevêt*" belonging to this "*born enemy of truth*", that is to say, to the Académie des Sciences de Paris. No doubt Wroński's clear-cut opinion would nowadays be totally reversed, now that algorithmics and combinatorics are so well received in this noble assembly [5]; he would no more write that *the only aim of this corporation is exploitation of Man, consequently exploitation of Heads of State, using the imposing authority of Science* [10, p. 4].

Thus, instead of devoting his full attention to solving the following *rigorous system of equations* [11, p. 6]:

> *"Let α be the anarchy degree, δ the degree of despotism. Then one has the following precise relations:*

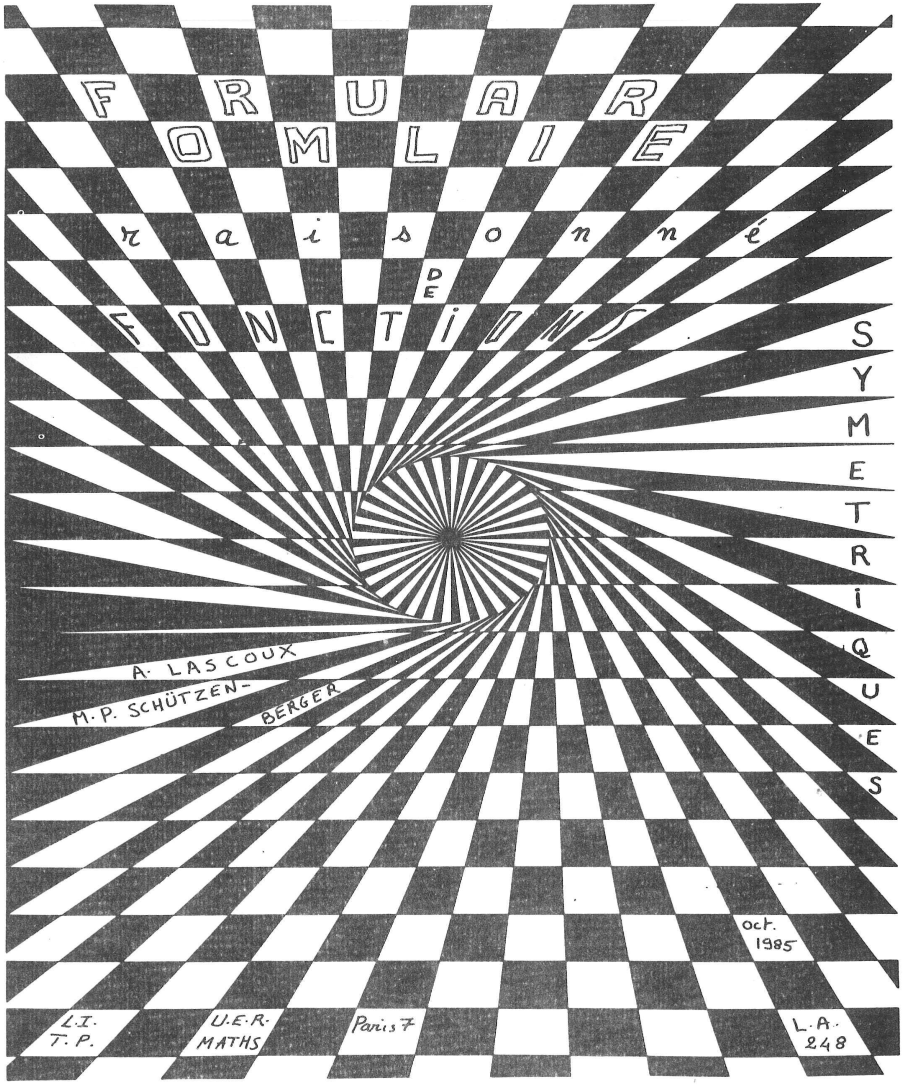
$$\alpha = \left\{\frac{m+n}{m}\cdot\frac{m+n}{n}\right\}^{p-r} \times \left(\frac{m}{n}\right)^{p+r}, \tag{1.1}$$

$$\delta = \left\{\frac{m+n}{m}\cdot\frac{m+n}{n}\right\}^{r-p} \times \left(\frac{n}{m}\right)^{p+r}, \tag{1.2}$$

> *where m represents the numerical influence of the national party, p the standard deviation of the philosophy of this party from true religion, n the influence of the moral party and r the deviation of religion from true philosophy".*

Wroński had to write such trivial things as the *Résolution Générale des Equations (de tout degré)* which we shall examine in detail in Section 2. For a survey of his mathematical work, we refer to [1].

> *"We do not need to emphasize how painful such a pedestrian task must be to a man who, in the innermost recesses of his retreat, has spent his life scrutinizing and discovering creation laws, as well as the final destiny of rational beings"* [10, p. 14].



Les pages qui suivent sont les notes du cours d'informatique ayant le même intitulé.

A l'heure où malgrè les peurs et les incertitudes, le succès des ordinateurs marque notre époque de son sot, il n'est point besoin de commenter l'importance pour l'action du sujet dont on n'a fait ici qu'effleurer les aspects les plus immédiatement applicables ; c'est ce qu'on compris les étudiants-chercheurs et les chercheurs enseignants qui ont participé nombreux à cette entreprise pédagogique et dont les critiques et suggestions ont été ci-incorporés avec notre gratitude.

Le monoide plaxique, les q-fonctions symétriques et les polynômes de Schubert ont une place de choix dans cet avenir républicain (mais aussi écologique) et seront l'objet de 8 (ou 9) fascicules ultérieurs.

Nous remercions Monsieur le Directeur de l'U.E.R. R.Godement sous l'égide duquel les autorisations nécessaires nous ont été accordées dans le cadre scientifique de l'Université Paris 7 (Président, Prof. Le Fol) .

Le L.I.T.P., L.A.248, et son Président, le Professeur Nivat, sont toutefois seuls responsables du contenu technique et de l'orientation volontairement donnée à cet exposé qui s'appuye sur les travaux classiques des algébristes des siècles passés.

Nos références essentielles sont :

Th. MUIR — The theory of determinants in the Historical order, cinq volumes chez Macmillan dont la publication s'étale entre 1906-1928 ; réimpression par Dover, 1960.

D.E.LITTLEWOOD — The Theory of Group Characters, Oxford 1950

I.G.MACDONALD — Symmetric Functions and Hall Polynomials, Oxford Math.Mono 1979.

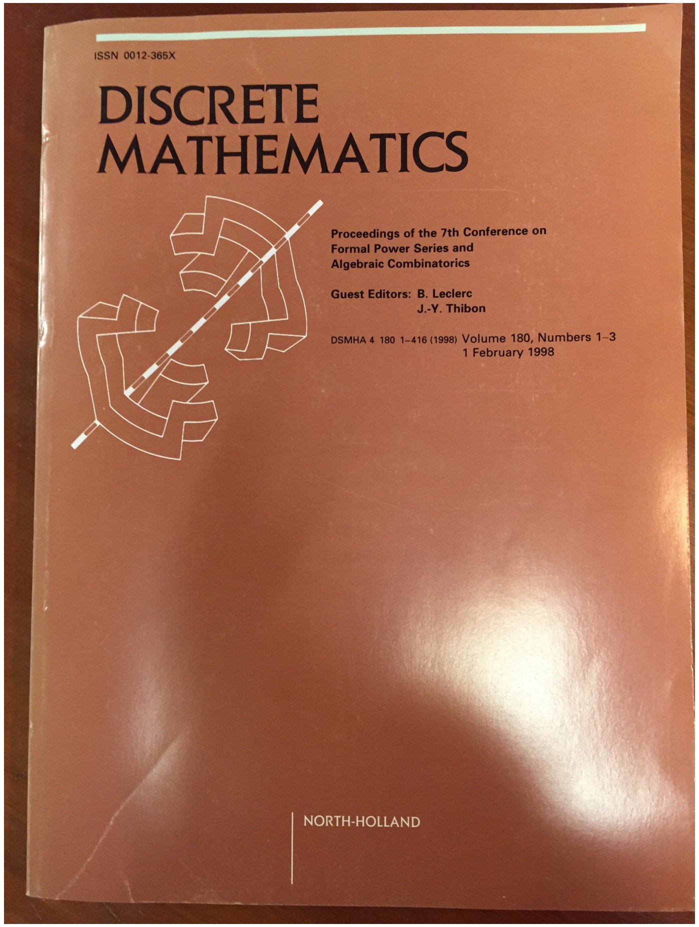
Saint Mélany 1991

The Phalanstère (Florence 1993)

Marne-la-Vallée 1993

Saint Mélany 1994

1995: FPSAC, LLT polynomials



Lascoux, Leclerc, Thibon, SLC34g https://www.mat.univie.ac.at/~slc/wpapers/s34...

Séminaire Lotharingien de Combinatoire, B34g (1995), 23pp.

Alain Lascoux, Bernard Leclerc and Jean-Yves Thibon

Ribbon tableaux, Hall-Littlewood functions and unipotent varieties

Abstract. We introduce a new family of symmetric functions, which are defined in terms of ribbon tableaux and generalize Hall-Littlewood functions. We present a series of conjectures, and prove them in two special cases.

The following versions are available:

- PDF (not all figures correctly displayed) (295 K)
- PostScript (342 K)
- dvi version (not all figures correctly displayed)
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Original motivation: plethysm

- Irreducible tensor representations of $GL(n, \mathbb{C})$:

$$\rho_\lambda : \ GL(n, \mathbb{C}) \longrightarrow GL(V_\lambda), \quad V_\lambda \subseteq (\mathbb{C}^n)^{\otimes k}$$

- λ partition of k with at most n parts
- Character: Schur function $s_\lambda = \mathrm{ch}(\rho_\lambda)$
- Composition of two representations ρ of character f and η of character g:

$$\mathrm{ch}(\eta \circ \rho) =: g \circ f \quad \text{plethysm of } f \text{ by } g\text{, also denoted by } g[f]$$

- The problem: compute

$$s_\lambda[s_\mu] = \sum_\nu d^\nu_{\lambda\mu} s_\nu$$

- More precisely, find a *combinatorial* description
- if $\lambda \vdash d$, $s_\lambda[s_\mu]$ is a part of

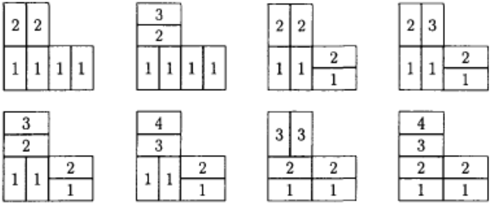
$$s_\mu^d = \sum_{\nu \vdash nd} c^\nu_{\mu\mu\cdots\mu} s_\nu = \sum_{\lambda \vdash d} f^\lambda s_\lambda[s_\mu]$$

  where $c^\nu_{\mu\mu\cdots\mu}$ are the Littlewood-Richardson coefficients, and f^λ the number of standard tableaux of shape λ.
- For $d = 2$, no multiplicities

$$V \otimes V = S^2(V) \oplus \Lambda^2(V) \Leftrightarrow s_\mu^2 = h_2[s_\mu] + e_2[s_\mu]$$

- First problem: split the Littlewood-Richardson tableaux into two sets, corresponding to the symmetric and antisymmetric parts of the square.
- Idea (B.L.) Formulate a version of the LR-rule with domino tableaux, and split according to the parity of half the number of horizontal dominos.

$$s_{21}^2 = s_{42} + s_{411} + s_{33} + 2s_{321} + s_{3111} + s_{222} + s_{2211}$$



$$\begin{cases} h_2[s_{21}] & = s_{42} + s_{321} + s_{3111} + s_{222} \\ e_2[s_{21}] & = s_{411} + s_{33} + s_{321} + s_{2211} \end{cases}$$

[C. Carré, B. Leclerc, Séminaire Lotharingien de Combinatoire, B31c (1993), 8 pp; J. Alg.Combin. **4** (1995), 201–231]

What about higher powers?

Next step suggested by previous LLT results on Hall-Littlewood functions at roots of unity

- Hall-Littlewood functions

$$P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu}(t) P_\lambda$$

  such that $g^\lambda_{\mu\nu}(q) = q^{n(\lambda)-n(\mu)-n(\nu)} f^\lambda_{\mu\nu}(q^{-1})$ (Hall algebra)
- Kostka numbers

$$s_\lambda = \sum_\mu K_{\lambda\mu}(t) P_\mu$$

- Kostka numbers are special LR coefficients

$$K_{\lambda\mu} = c^\lambda_{\mu_1,\mu_2,\dots,\mu_r}$$

- Dual HL functions

$$\langle Q'_\mu, P_\nu \rangle = \delta_{\mu\nu} \quad (\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu})$$

are t-analogues of products h_μ

$$Q'_\mu = \sum_\lambda K_{\lambda\mu}(t) s_\lambda \longrightarrow h_\mu \ (t \to 1)$$

- The Kostka-Foulkes polynomials $K_{\lambda\mu}(t) \in \mathbb{N}[t]$
- The $\tilde{K}_{\lambda\mu}(q)$ are (parabolic) Kazhdan-Lusztig polynomials for the affine symmetric group

Roots of unity and plethysm formulae

- $t = 1$ is not the only interesting value
- For $t = \zeta$ a primitive rth root of unity

$$Q'_\lambda(X;\zeta) = Q'_\mu(X;\zeta) \prod_{i\geq 1} \left[Q'_{(i^r)}(X;\zeta)\right]^{q_i}$$

where $\lambda = (1^{m_1} 2^{m_2} \dots n^{m_n})$, $m_i = rq_i + r_i$ with $0 \leq r_i < r$, and $\mu = (1^{r_1} 2^{r_2} \dots n^{r_n})$.

- and for rectangular partitions, we obtain plethysms with power-sums

$$Q'_{(n^r)}(X;\zeta) = (-1)^{(r-1)n} p_r[h_n(X)]$$

Interpretation

Consider the (reducible) $GL(n,\mathbb{C})$-module

$$V = \Lambda^{\nu_1}\mathbb{C}^n \otimes \Lambda^{\nu_2}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{\nu_r}\mathbb{C}^n$$

and the cyclic shift operator $\gamma : \; V^{\otimes d} \mapsto V^{\otimes d}$

$$\gamma(v_1 \otimes v_2 \otimes \cdots \otimes v_d) = v_d \otimes v_1 \otimes \cdots \otimes v_{d-1}$$

Its eigenspaces $W^{(k)}$ are representations of $GL(n,\mathbb{C})$.
The previous formulae imply a combinatorial description of their characters $\ell_d^{(k)}[e_\nu]$.

Can we do the same starting with $V = V_\lambda$ irreducible ?

Answer: ribbon tableaux

Ribbon tableaux and products of Schur functions

A Schur function $s_\lambda(X)$ is a sum over semi-standard Young tableaux t of shape λ

$$s_\lambda(X) = \sum_{t \in \mathrm{Tab}\,(\lambda)} X^t$$

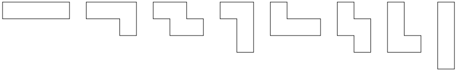
where $X^t = \prod_i x_i^{m_i}$, m_i number of occurences of i in t.
A product of r Schur functions $s_{\mu^{(i)}}$ is a sum over r-tuples of tableaux

$$s_{\mu^{(1)}} s_{\mu^{(2)}} \cdots s_{\mu^{(r)}} = \sum_{(t_1,\dots,t_r)} X^{t_1} X^{t_2} \cdots X^{t_r}$$

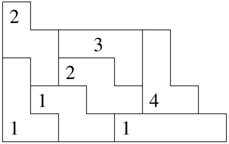
r-tuples of tableaux $\longleftrightarrow$ r-ribbon tableaux

Ribbons (rim-hooks) and ribbon tableaux

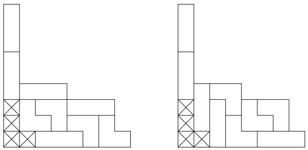
Here are the ($2^3 = 8$) 4-ribbons



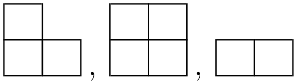
and a 4-ribbon tableau of shape (87661) and weight (3211)



The partition $\lambda = (87^2 41^5)$ has as 3-core $\nu = (211)$



and as 3-quotient the triple $((21), (22), (2))$

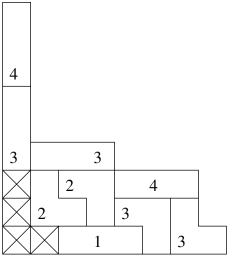


The Stanton-White bijection

Choosing as 3-core $\kappa = (211)$, the triple

$$\begin{array}{|c|c}\hline 4 & \multicolumn{1}{c}{} \\ \hline 3 & \multicolumn{1}{c|}{3} \\ \hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}$$

with weights $(0021), (1111), (0110)$ corresponds to the 3-ribbon tableau of shape $\lambda = (87^2 41^5)$ and weight $\mu = (1242)$.



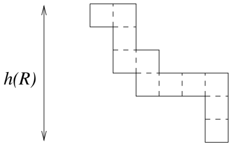
If μ is the partition with r-quotient $(\mu^{(0)}, \ldots, \mu^{(r-1)})$ and empty r-core

$$s_{\mu^{(0)}} s_{\mu^{(1)}} \cdots s_{\mu^{(r-1)}} = \sum_{T \in \mathrm{Tab}_r(\mu,\cdot)} X^T$$

where $\mathrm{Tab}_r(\mu, \cdot)$ is the set of r-ribbon tableaux of shape μ

A natural statistic on ribbon tableaux is the sum of the heights of the ribbons

Example: $r = 11$, $h(R) = 6$



Spin and cospin

The relevant statistic is rather $h(R) - 1$, and for compatiblity with Hall-Littlewood functions, one introduces the *spin*

$$s(R) = \frac{1}{2}(h(R) - 1), \quad s(T) = \sum_{R \in T} s(R)$$

(a half-integer in general) and the *cospin* (an integer)

$$\tilde{s}(T) = s_r^*(\mu) - s(T) \quad \text{for } T \in \mathrm{Tab}_r(\mu, \cdot)$$

The most general q-LR coefficients are defined by

$$\tilde{G}_\mu = \sum_{T \in \mathrm{Tab}_r(\mu, \cdot)} q^{\tilde{s}(T)} X^T = \sum_\lambda c^\lambda_{\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(r-1)}}(q) s_\lambda(X)$$

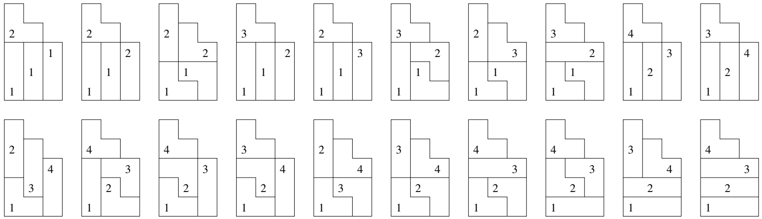
The 3-quotient of $\lambda = (33321)$ is $((1), (1,1), (1))$ and the q-analogue of $s_1 s_{11} s_1$ (in this order) is

$$m_{31} + (1+q)m_{22} + (2+2q+q^2)m_{211} + (3+5q+3q^2+q^3)m_{1111}$$

$$= (s_{31} - s_{22} - s_{211} + 2\, s_{1111}) + (1+q)(s_{22} - s_{211} + s_{1111})$$
$$+ (2+2q+q^2)(s_{211} - 3\, s_{1111}) + (3+5q+3q^2+q^3)s_{1111}$$
$$= s_{31} + qs_{22} + (q+q^2)s_{211} + q^3 s_{1111}$$

The $c^{\lambda}_{\mu_1,\mu_2,\ldots,\mu_r}(q)$ are defined by an alternating sum but are in $\mathbb{N}[q]$.

The monomial expansion above is given by the 3-ribbon tableaux of shape (33321) and dominant weight



The H-functions

- Family of spin t-analogues related to HL functions.
- A partition of the form $\lambda = r\mu = (r\mu_1, \ldots, r\mu_s)$ has empty r-core
- Its r-quotient is obtained by grouping the parts of μ according to their class modulo r

$$\lambda(i) = \{\mu_j | j \equiv -i \mod r\}$$

- For any r, the symmetric functions

$$H_\mu^{(r)}(X; t) = \sum_{T \in \mathrm{Tab}_r(r\mu, \cdot)} t^{s(T)} X^T$$

form a basis which is unitriangular on Schur functions
- It can be proved that for $r \geq \ell(\mu)$,

$$H_\mu^{(r)}(X; t) = Q'_\mu(X; t)$$

Some conjectures for H-functions

- **Monotonicity** $H_\mu^{(r+1)} - H_\mu^{(r)}$ is positive on the Schur basis, that is, the coefficients are in $\mathbb{N}[t]$.
- **Plethysm** When $\mu = \nu^r$, for ζ a primitive r-th root of unity,

$$H_{\nu^r}^{(r)}(\zeta) = (-1)^{(r-1)|\nu|}\, p_r[s_\nu]$$

  and when $d|r$ and ζ is a primitive d-th root of unity,

$$H_{\nu^r}^{(r)}(\zeta) = (-1)^{(d-1)|\nu|r/d} p_d^{r/d}[s_\nu]\ .$$

- Equivalently,

$$H_{\nu^r}^{(r)}(t) \mod 1 - t^r = \sum_{i=0}^{r-1} t^i \ell_r^{(i)}[s_\nu]$$

- **Proved by Kazuto Iijima** [European J. Combin. **34** (2013) 968–986]

Examples

The H-functions associated with the partition $\lambda = (3211)$ are

$$\begin{aligned}
H^{(2)}_{3211} &= s_{3211} + t\, s_{322} + t\, s_{331} + t\, s_{4111} \\
&\quad + (t + t^2)\, s_{421} + t^2\, s_{43} + t^2\, s_{511} + t^3\, s_{52} \\
H^{(3)}_{3211} &= s_{3211} + t\, s_{322} + (t + t^2)\, s_{331} + t\, s_{4111} \\
&\quad + (t + 2t^2)\, s_{421} + (t^2 + t^3)\, s_{43} + (t^2 + t^3)\, s_{511} \\
&\quad + 2t^3\, s_{52} + t^4\, s_{61} \\
H^{(4)}_{3211} &= s_{3211} + t\, s_{322} + (t + t^2)\, s_{331} + t\, s_{4111} + (t + 2t^2 + t^3)\, s_{421} \\
&\quad + (t^2 + t^3 + t^4)\, s_{43} + (t^2 + t^3 + t^4)\, s_{511} \\
&\quad + (2t^3 + t^4 + t^5)\, s_{52} + (t^4 + t^5 + t^6)\, s_{61} + t^7\, s_7 \\
&= Q'_{3211}
\end{aligned}$$

The plethysms of s_{21} with the cyclic characters $\ell_3^{(i)}$ are given by the reduction modulo $1 - t^3$ of $H^{(3)}_{222111}$

$$\begin{aligned}
H^{(3)}_{222111} &= t^9 s_{63} + (t+1)t^7 s_{621} + t^6 s_{6111} + (t+1)t^7 s_{54} \\
&+ (t^3+2t^2+2t+1)t^5 s_{531} + (t^2+2t+1)t^5 s_{522} \\
&+ (t^3+2t^2+2t+1)t^4 s_{5211} + (t+1)t^4 s_{51111} \\
&+ (t^2+2t+1)t^5 s_{441} + (t^3+2t^2+3t+2)t^4 s_{432} \\
&+ (2t^3+3t^2+3t+1)t^3 s_{4311} + (t^3+3t^2+3t+2)t^3 s_{4221} \\
&+ (t^3+2t^2+2t+1)t^2 s_{42111} + t^3 s_{411111} + (t^3+1)t^3 s_{333} \\
&+ (2t^3+3t^2+2t+1)t^2 s_{3321} + (t^2+2t+1)t^2 s_{33111} \\
&+ (t^2+2t+1)t^2 s_{3222} + (t^3+2t^2+2t+1)t s_{32211} \\
&+ (t+1)t s_{321111} + (t+1)t s_{22221} + s_{222111}
\end{aligned}$$

$$\ell_3^{(0)} = s_3 + s_{111}$$
$$\ell_3^{(1)} = s_{21}$$
$$\ell_3^{(2)} = s_{21}$$

In general,

$$\ell_n^{(k)} = \sum_{\substack{t \in \mathrm{STab}\,(n) \\ \mathrm{maj}(t) \equiv k \mod n}} s_{\mathrm{shape}(t)}$$

$$\begin{aligned}
(h_3 + e_3)[s_{21}] &= s_{222111} + 2s_{33111} + 3s_{4311} + 2s_{32211} + 2s_{42111} \\
&+ 3s_{4221} + 2s_{3222} + 2s_{3321} + s_{411111} \\
&+ 2s_{333} + s_{6111} + 2s_{531} + 2s_{5211} + 2s_{432} \\
s_{21}[s_{21}] &= s_{3222} + 3s_{3321} + 2s_{32211} + 2s_{42111} + s_{22221} + s_{33111} \\
&+ s_{321111} + 3s_{4311} + 3s_{4221} + s_{441} + s_{522} \\
&+ 2s_{5211} + s_{51111} + 2s_{531} + 3s_{432} + s_{621} + s_{54}
\end{aligned}$$

Ribbons tableaux and the Fock space

- The algebra of symmetric functions can be identified with the Fock space representation of $\widehat{\mathfrak{gl}}_\infty$.

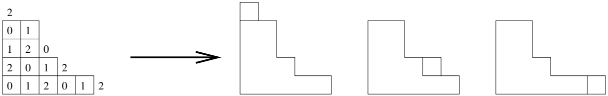
$$s_\lambda \leftrightarrow |\lambda\rangle = v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots \quad \text{where } i_k = \lambda_k - k + 1$$

- This induces actions of $\widehat{\mathfrak{gl}}_r = \widehat{\mathfrak{sl}}_r + \mathcal{H}_r$ where $\mathcal{H}_r$ is a Heisenberg algebra
- Bosonic Fock space $\mathcal{F} = \mathbb{C}[x_1, x_2, \ldots] \simeq Sym\ (x_k = \frac{1}{k} p_k)$
- Action of $\widehat{\mathfrak{gl}}_r$ on $\mathcal{F}$:
  - the generator B_k of $\mathcal{H}_r$ acts by $rk\frac{\partial}{\partial p_{rk}}$ for $k > 0$ and as the multiplication by p_{-rk} for $k < 0$.
  - Action of the generators of $\widehat{\mathfrak{sl}}_r$ particularly simple in the basis of Schur functions s_λ.

For a node γ in ith row and jth column of λ let $r(\gamma) = j - i \mod r$.
Then,

$$e_i s_\lambda = \sum s_\nu \,, \qquad f_i s_\lambda = \sum s_\mu \,,$$

where ν (resp. μ) runs through all partitions obtained from λ by removing (resp. adding) a node of residue i.
For example, f_2 of $\widehat{\mathfrak{s}l}_3$ acts on s_{5322} by



- $U(\mathcal{H}_r) = p_r \circ Sym$ is as well generated by the

$$V_k = \text{ `multiplication by } p_r \circ h_k\text{'}$$

$$V_k s_\lambda = \sum (-1)^{\mathbf{h}(\mu/\lambda)} s_\mu$$

sum over all partitions μ such that μ/λ is a horizontal r-ribbon strip of weight k, where

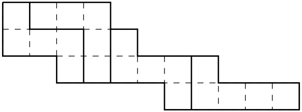
$$\mathbf{h}(\mu/\lambda) = \sum_R (h(R) - 1)$$

sum over all the r-ribbons R tiling μ/λ.

- and their adjoints U_k

$$U_k s_\mu = \sum (-1)^{\mathbf{h}(\mu/\lambda)} s_\lambda$$

sum over all partitions λ such that μ/λ is a horizontal r-ribbon strip of weight k.



A horizontal 5-ribbon strip of weight 4 and spin $\frac{7}{2}$

- In the $\mathbb{Q}(q)$-vector space

$$\mathcal{F} = \bigoplus_{\lambda \in \mathbf{P}} \mathbb{Q}(q)\,|\lambda\rangle$$

  $\gamma = (a, b) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ is an indent i-node of λ if a box of residue $i = a - b \mod r$ can be added to λ at position (a, b)
- Similarly, a node of residue i which can be removed is called a removable i-node.
- $i \in \{0, 1, \ldots, r-1\}$
- λ, ν such that $\nu/\lambda = \gamma = \boxed{i}$

Defining some numbers associated with a partition

- $N_i(\lambda) = \sharp\{$ indent i-nodes of λ $\} - \sharp\{$ removable i-nodes of λ $\}$,
- $N_i^l(\lambda,\nu) = \sharp\{$ indent i-nodes of λ on the *left* of γ (not counting γ) $\}$ $-\sharp\{$ removable i-nodes of λ on the *left* of γ $\}$,
- $N_i^r(\lambda,\nu) = \sharp\{$ indent i-nodes of λ on the *right* of γ (not counting γ) $\}$ $-\sharp\{$ removable i-nodes of λ on the *right* of γ $\}$,
- $N^0(\lambda) = \sharp\{$ 0-nodes of λ $\}$.

One can construct q-analogues of the previous representations

iwa 1990

$$f_i|\lambda\rangle = \sum_\mu q^{N_i^r(\lambda,\mu)}|\mu\rangle\,, \qquad e_i|\mu\rangle = \sum_\lambda q^{N_i^l(\lambda,\mu)}|\lambda\rangle$$

$$q^{h_i}|\lambda\rangle = q^{N_i(\lambda)}|\lambda\rangle \quad \text{and} \quad q^D|\lambda\rangle = q^{-N^0(\lambda)}|\lambda\rangle$$

defines an action of $U_q(\widehat{\mathfrak{sl}}_r)$

- Can be extended to $U_q(\widehat{\mathfrak{gl}}_r)$ (q-wedges and q-bosons of [Kashiwara-Miwa-Stern 1996].)
- Key point: 'q-bosons' B_k can be replaced by q-analogues of U_k and V_k

$$V_k|\lambda\rangle = \sum(-q)^{-\mathbf{h}(\mu/\lambda)}|\mu\rangle \qquad U_k|\mu\rangle = \sum(-q)^{-\mathbf{h}(\mu/\lambda)}|\lambda\rangle$$

- The relations $[U_i, U_j] = [V_i, V_j] = 0$ prove that the H-functions are symmetric (more elementary proofs since then)

- Identify $\mathcal{F}_q \simeq \mathbb{Q}(q) \otimes Sym$ by $|\lambda\rangle = s_\lambda$
- Define a linear operator $\psi_q^r : \mathcal{F}_q \longrightarrow \mathcal{F}_q$ by

$$\psi_q^r(h_\lambda) = V_{\lambda_1} V_{\lambda_2} \cdots V_{\lambda_r} |\emptyset\rangle$$

- Then,

$$\psi_q^n(h_\mu) = \sum_{T \in \mathrm{tab}\, _r(\,\cdot\,,\mu)} (-q)^{-2s(T)} s_{\mathrm{shape}(T)}$$

- The image $\{\psi_q^r(g_\lambda)\}$ of any basis $\{g_\lambda\}$ is a basis of the space of $U_q(\widehat{\mathfrak{s}l}_r)$-highest weight vectors in $\mathcal{F}_q$.
- Taking $g_\lambda = s_\lambda$, we have

$$\langle \psi_q^r(s_\lambda), s_\mu \rangle = (-q)^{2s_r^*(\mu)} c^\lambda_{\mu^{(0)}\ldots,\mu^{(r-1)}}(q^2)$$

$((\mu^{(0)} \ldots, \mu^{(r-1)})$ r-quotient of $\mu)$.

Canonical bases

- As an $U_q(\widehat{\mathfrak{gl}}_r)$-module, $\mathcal{F}_q$ is irreducible.
- But as $U_q(\widehat{\mathfrak{sl}}_r)$-module,

$$\mathcal{F}_q \simeq \bigoplus_{m\geq 0} L(\Lambda_0 - m\delta)^{\oplus p(m)}$$

- Each simple $U_q(\widehat{\mathfrak{sl}}_r)$-module $L(\Lambda_0 - m\delta)$ has a canonical basis but these cannot be pieced together to form a canonical basis of the whole $\mathcal{F}_q$ under $U_q(\widehat{\mathfrak{gl}}_r)$.
- Such a basis (G^-_λ) was defined in [Leclerc-T. 1996].
- All the q-plethysms $\psi^r_q(s_\nu)$ are members of this basis.
- The coefficients of the dual basis on Schur functions were conjectured to give the decomposition matrices of quantized Schur algebras at roots of unity.

- The proof of this conjecture [Varagnolo-Vasserot 1999] allows one to identify the q-LR coefficients with parabolic KL polynomials [Leclerc-T. 2000]
- Then, a result of [Kashiwara-Tanisaki 1999] shows that $c^{\lambda}_{\mu^{(0)},\dots,\mu^{(r-1)}}(q) \in \mathbb{N}[q]$
- A combinatorial proof is still wanted for general r.
- Combinatorial formula for $r = 3$ [J. Blasiak, Math. Z. **283** (2016), 601–628]
- LLT polynomials have been defined for other root systems by Lecouvey [European J. of Combin. **30** (2009) 157–191], and Grojnowski-Haiman (unpublished)
- In both versions, the coefficients are parabolic KL polynomials

Upper and lower canonical bases of $\mathcal{F}_q$

- There is a unique q-semi-linear endomorphism $x \mapsto \bar{x}$ of $\mathcal{F}_q$ such that $\overline{|\emptyset\rangle} = |\emptyset\rangle$, $\overline{f_i x} = f_i \bar{x}$ and $\overline{V_k x} = V_k \bar{x}$.
- In terms of q-wedges, reverse a prefix and normalize

$$|\lambda\rangle = u_I = u_{i_1} \wedge_q u_{i_2} \wedge_q \cdots u_{i_m} \wedge_q \cdots$$

$$\overline{u_I} = (-1)^{\binom{k}{2}} q^{\alpha_{n,k}(I)} u_{i_k} \wedge_q u_{i_{k-1}} \wedge_q \cdots \wedge_q u_{i_1} \wedge_q u_{i_{k+1}} \wedge_q u_{i_{k+2}} \wedge_q \cdots$$

- Let

$$\mathcal{L}^+ = \bigoplus_\lambda \mathbb{Z}[q]|\lambda\rangle \quad \text{and} \quad \mathcal{L}^- = \bigoplus_\lambda \mathbb{Z}[q^{-1}]|\lambda\rangle$$

- There exists bases G_λ^+ and G_λ^- of $\mathcal{F}_q$ characterized by

  (i) $\overline{G_\lambda^+} = G_\lambda^+$, $\overline{G_\lambda^-} = G_\lambda^-$
  (ii) $G_\lambda^+ \equiv |\lambda\rangle \mod q\mathcal{L}^+$, $G_\lambda^- \equiv |\lambda\rangle \mod q^{-1}\mathcal{L}^-$

- Let

$$G^+_\mu = \sum_\lambda d_{\lambda\mu}(q)|\lambda\rangle$$

and

$$G^-_\lambda = \sum_\mu e_{\lambda\mu}(-q^{-1})|\mu\rangle$$

- Then,

$$e_{\lambda\mu}(q) = \sum_{x\in\widehat{\mathfrak{S}}(a)} (-q)^{\ell(x)} P_{w_v x, w_u}(q)$$

$$d_{\lambda\mu}(q) = \sum_{y\in\mathfrak{S}_m} (-q)^{\ell(y)} P_{y\hat{w}_u, \hat{w}_v}(q)$$

(parabolic KL polynomials of Deodhar).

Quantized Schur algebras at roots of 1

- $\mathcal{S}_n(\zeta)$ with ζ a primitive r-th root of 1
- $W(\lambda)$ Weyl modules. $L(\mu)$ simple modules
- **Conjecture** [LLT] let $\{W(\lambda)^i\}$ be the Jantzen filtration

$$d_{\lambda'\mu'}(q) = \sum_{i\geq 0}[W(\lambda)^i/W(\lambda)^{(i+1)} : L(\mu)]q^i$$

- Extends the LLT conjecture proved by Ariki.
- Proved by Varagnolo-Vasserot for $q = 1$.
- **Proved by P. Shan** [Represent. Theory 16 (2012), 212-269] for $\zeta = e^{2i\pi/k}$, $k \leq -3$
- One has $[d_{\lambda\mu}(q)] = [e_{\lambda'\mu'}(-q)]^{-1}$.

Back to Hall-Littlewood functions

- Why do we have $\tilde{K}_{\lambda\mu}(q) = c^{\lambda}_{\mu_1,\ldots,\mu_r}(q)$?
- One can now deduce it from an earlier result of Lusztig

$$e_{N\lambda,N\mu}(q) = \tilde{K}_{\lambda\mu}(q^2) \quad (N \geq m)$$

- Original proof [LLT97]: cell decompositions of unipotent varieties
- Open problem: similar interpretation for other LLT polynomials ?
- Cospin q-analogues $\tilde{G}_{\mu}(X; 1+q)$ of products of arbitrary vertical strips are e-positive [P. Alexandersson, arXiv:1903.03998; M. d'Adderio, JCTA 172 (2020)],
- Not true in general. Known for $\tilde{Q}'_{\mu}(X; 1+q)$, special case of a property of Hall polynomials

Unipotent varieties

- The coefficients $\tilde{g}_{\nu\mu}(q)$ of the monomial expansions

$$\tilde{Q}'_\mu(X;q) := \sum_\lambda \tilde{K}_{\lambda\mu}(q) s_\lambda = \sum_\nu \tilde{g}_{\nu\mu}(q) m_\nu$$

  are the Poincaré polynomials of certain algebraic varieties.
- Let $u \in GL(n,\mathbb{C})$ be a unipotent element of Jordan type μ, and let $\mathcal{F}_\nu$ be the variety of ν-flags in $V = \mathbb{C}^n$

$$V_{\nu_1} \subset V_{\nu_1+\nu_2} \subset \ldots \subset V_{\nu_1+\ldots\nu_r} = V$$

  where $\dim V_i = i$.
- The unipotent variety $\mathcal{F}_\nu^u$ is the set of fixed points of u in $\mathcal{F}_\nu$.

Cell decompositions

- Cell decomposition of $\mathcal{F}_\nu^u$ involving only cells of even real dimensions $\simeq \mathbb{C}^d$ [Shimomura 1980].
- Hence, the Poincaré polynomial has the form

$$\Pi_{\nu\mu}(t^2) = \sum_i t^{2i} \dim H_{2i}(\mathcal{F}_\nu^u, \mathbb{Z})$$

and $\Pi_{\nu\mu}(q) = |\mathcal{F}_\nu^u[\mathbb{F}_q]|$, which can be shown (by means of the Hall algebra) to be

$$|\mathcal{F}_\nu^u[\mathbb{F}_q]| = \tilde{g}_{\nu\mu}(q)$$

- Cells are parametrized by *tabloids*.

Tabloids

- For μ, ν arbitrary compositions of n, a μ-tabloid of shape ν is a filling of the diagram with row lengths $\nu_1, \nu_2, \ldots, \nu_r$ such that i occurs μ_i times, each row nondecreasing.
- For example,

$$\begin{array}{|c|c|c|}
\cline{1-1}
3 \\
\hline
1 & 1 & 1 \\
\hline
1 & 1 & 3 \\
\hline
2 & 3 \\
\cline{1-2}
\end{array}$$

is a $(5, 1, 3)$-tabloid of shape $(2, 3, 3, 1)$

Inversion statistic on tabloids

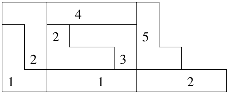
- Dimension $d(\mathbf{t})$ of the cell $c_{\mathbf{t}}$ explicitly given by Shimomura.
- A slightly modified version $e(\mathbf{t})$ (having the same distribution) can be interpreted as a kind of 'inversion number' on r-tuple of rows (e-inversions) [Terada 1993]
- Tabloid $\mathbf{t} = (w_1, \ldots, w_r) \simeq r$-tuple of row tableaux.
- y the k-th letter of w_i
- x the k-th letter of w_j
- For $x < y$ (y, x) is an e-inversion if either (a) $i < j$ or (b) $i > j$ and there is on the right of x in w_j a letter $u < y$
- $e(\mathbf{t})$ is equal to the number of inversions (y, x) in $\mathbf{t}$.

Inversions and cospin

Stanton-White correspondence maps $\mathbf{t}$ to T such that $\tilde{s}(T) = e(\mathbf{t})$ For example,

$$\mathbf{t} \quad = \quad \left(\begin{array}{c} \boxed{2}\boxed{3} \\ 1 \quad 1 \end{array} , \begin{array}{c} \boxed{1}\boxed{1}\boxed{2} \\ 0 \quad 0 \quad 0 \end{array} , \begin{array}{c} \boxed{4}\boxed{5} \\ 3 \quad 1 \end{array} , \begin{array}{c} \boxed{2} \\ 1 \end{array} \right)$$

has $e(T) = 7$ and is mapped to



of cospin 7.

Recent progress

- The generalized inversion number $e(\mathbf{t})$ has been extended to arbitrary r-tuples of tableaux [Schilling-Shimozono-White, Adv. Applied Math. **30** (2003) 258–272]
- Another version working with tuples of skew tableaux has been found by Haglund,Haiman, and Loehr [J. Amer. Math. Soc. **18** (2005), 735–761]
- It allowed these authors to prove the Schur positivity of Macdonald polynomials $\tilde{H}_\mu(x; q, t)$ by expressing them as $\mathbb{N}[q^{-1}, t]$ linear combination of special LLT polynomials
- These special polynomials are q-analogues of products of ribbon Schur functions
- The proof uses quasi-symmetric functions
- This suggests connections with noncommutative symmetric functions and combinatorial Hopf algebras

- Haglund and Wilson [arXiv:1701.05622]: Macdonald's $J_\mu(x; q, t)$ in terms of the quasi-symmetric chromatic polynomials [Shareshian-Wachs] of certain graphs
- Here, these chromatic polynomials are symmetric
- They are related to unicellular LLT-polynomials (t-analogues of s_1^n given by tuples of skew partitions with a single box) by

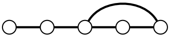
$$\mathrm{X}_G(t, X) = (t-1)^{-n}\,\mathrm{LLT}_G(t, (t-1)X)$$

[Carlsson and Mellit, J. Amer. Math. Soc. 31 (2018), 661–697]

Dyck graphs

- The graphs G are simple graphs with vertices labelled $1, \ldots, n$, such that if there is an edge (i, j) with $i < j$, then all the (i', j') with $i \leq i' < j' \leq j$ are also edges of G.
- The number of such graphs is the Catalan number c_n.
- Encoding by partitions contained in a staircase

$\times$	$\times$			5
$\times$	$\times$		4	
$\times$		3		
	2			
1				



- A coloring is proper if $c_i \neq c_j$ whenever $(i,j) \in E(G)$. We denote by $C(G)$ the set of proper colorings of G.
- The chromatic quasi-symmetric function of G expands in the M basis of $QSym$

$$\mathrm{X}_G(t,X) = \sum_{c \in C(G)} t^{\mathrm{asc}_G(c)} x_{c_1} x_{c_2} \cdots x_{c_n} = \sum_{c \in \mathrm{PC}(G)} t^{\mathrm{asc}_G(c)} M_{\mathrm{Ev}(c)}(X),$$

where $\mathrm{PC}(G)$ denotes the set of proper packed colorings, $\mathrm{asc}_G(c)$ is the number of edges $(i<j)$ such that $c_i < c_j$, and $\mathrm{Ev}(c)$ is the evaluation of c.

- $A = \{a_1 < a_2 < a_3 < \cdots\}$ totally ordered alphabet
- **WQSym**: "Word Quasi-Symmetric functions"

$$\mathbf{M}_u = \sum_{\mathrm{pack}(w)=u} w$$

$$\mathbf{M}_{121} = aba + aca + ada + bcb + bdb + cdc + \cdots$$

- Algebra:

$$\mathbf{M}_{u'}\mathbf{M}_{u''} = \sum_{\substack{u=vw \\ \mathrm{pack}(v)=u',\ \mathrm{pack}(w)=u''}} \mathbf{M}_u$$

- Hopf algebra $\Delta\mathbf{M}_u = \mathbf{M}_u(A \oplus B)$ (ordinal sum)
- Projection to $QSym$: $\mathbf{M}_u(X) = M_I(X)$

- The Guay-Paquet Hopf algebra $\mathcal{G}$: linear span of finite simple undirected graphs with vertices labelled by the first integers.
- Product: $G \cdot H = G \cup H[n]$ where $H[n]$ is H with labels shifted by the number n of vertices of G.
- Coproduct: G graph on n vertices, $w \in [r]^n$, coloring of G; $G|_w$ tensor product $G_1 \otimes \cdots \otimes G_r$ of the restrictions of G to vertices colored $1, 2, \ldots, r$.

$$\Delta^r G := \sum_{w \in [r]^n} t^{\mathrm{asc}_G(w)} G|_w. \tag{1}$$

- The subspace $\mathcal{D}$ of $\mathcal{G}$ spanned by Dyck graphs is a Hopf subalgebra.

- Given a Dyck graph G, define

$$\mathbf{X}_G(t,A) = \sum_{c\in \mathrm{PC}(G)} t^{\mathrm{asc}_G(c)} \mathbf{M}_c(A) \in \mathbf{WQSym}.$$

- Then, [Novelli, T., arXiv:1907.00077] $G \mapsto \mathbf{X}_G(A)$ is a morphism of Hopf algebras from $\mathcal{G}$ to **WQSym.**
- The $(1-t)$ transform and its inverse can be extended to **WQSym**
- Appliying it to $\mathbf{X}_G$, we find

$$(t-1)^n \mathbf{X}_G\left(t, \frac{A|}{|t-1}\right) = \sum_{u\in PW_n} t^{\mathrm{asc}_G(u)} \mathbf{M}_u(A).$$

The r.h.s. is therefore a noncommutative lift of the LLT polynomial LLT_G.

$$\mathbf{X}_{(\circ\quad\circ\quad\circ)} = \sum_{w \in PW(3)} \mathbf{M}_w$$

$$\begin{aligned}\mathbf{X}_{(\circ\!-\!\circ\quad\circ)} = {} & t\,\mathbf{M}_{121} + t\,\mathbf{M}_{122} + t\,\mathbf{M}_{123} + t\,\mathbf{M}_{132} + \mathbf{M}_{211} \\ & + \mathbf{M}_{212} + \mathbf{M}_{213} + t\,\mathbf{M}_{231} + \mathbf{M}_{312} + \mathbf{M}_{321}\end{aligned}$$

$$\begin{aligned}\mathbf{X}_{(\circ\quad\circ\!-\!\circ)} = {} & t\,\mathbf{M}_{112} + \mathbf{M}_{121} + t\,\mathbf{M}_{123} + \mathbf{M}_{132} + t\,\mathbf{M}_{212} \\ & + t\,\mathbf{M}_{213} + \mathbf{M}_{221} + \mathbf{M}_{231} + t\,\mathbf{M}_{312} + \mathbf{M}_{321}\end{aligned}$$

$$\begin{aligned}\mathbf{X}_{(\circ\!-\!\circ\!-\!\circ)} = {} & t\,\mathbf{M}_{121} + t^2\,\mathbf{M}_{123} + t\,\mathbf{M}_{132} + t\,\mathbf{M}_{212} \\ & + t\,\mathbf{M}_{213} + t\,\mathbf{M}_{231} + t\,\mathbf{M}_{312} + \mathbf{M}_{321}\end{aligned}$$

$$\mathbf{X}_{\left(\overset{\frown}{\circ\!-\!\circ\!-\!\circ}\right)} = t^3\,\mathbf{M}_{123} + t^2\,\mathbf{M}_{132} + t^2\,\mathbf{M}_{213} + t\,\mathbf{M}_{231} + t\,\mathbf{M}_{312} + \mathbf{M}_{321}$$

Analogue of F-positivity

$$\check{\Phi}_u = \sum_{v \geq \bar{u}} \mathbf{M}_{\bar{v}} \;\mapsto F_I(X)$$

$$\mathbf{LLT}_G = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{asc}_G(\sigma)} \check{\Phi}_{\mathrm{min}'_{G_\emptyset}(\sigma)}$$

where $G_\emptyset$ is the graph with n vertices and no edges.

$$\mathbf{LLT}_{(\circ \quad \circ \quad \circ)} = \check{\Phi}_{123} + \check{\Phi}_{122} + \check{\Phi}_{112} + \check{\Phi}_{121} + \check{\Phi}_{212} + \check{\Phi}_{111},$$

$$\mathbf{LLT}_{(\circ\!-\!\circ \quad \circ)} = t\,\check{\Phi}_{123} + t\,\check{\Phi}_{122} + \check{\Phi}_{112} + t\,\check{\Phi}_{121} + \check{\Phi}_{212} + \check{\Phi}_{111},$$

$$\mathbf{LLT}_{(\circ \quad \circ\!-\!\circ)} = t\,\check{\Phi}_{123} + \check{\Phi}_{122} + t\,\Phi_{112} + \check{\Phi}_{121} + t\,\check{\Phi}_{212} + \check{\Phi}_{111},$$

$$\mathbf{LLT}_{(\circ\!-\!\circ\!-\!\circ)} = t^2\,\check{\Phi}_{123} + t\,\check{\Phi}_{122} + t\,\check{\Phi}_{112} + t\,\check{\Phi}_{121} + t\,\check{\Phi}_{212} + \check{\Phi}_{111}.$$

A conjecture

Let $\hat{Q}'(X;t) = (1-t)^{-\ell(\mu)} Q'(X;t)$.
Spin-unicellular LLT

$$X_G(t) = (1-t)^{-n} LLT_G((1-t)X;t)$$

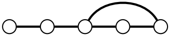
Define

$$(1-t)^{-n} LLT_G(X;t) = \sum_{\mu \vdash n} c_G^{\mu}(t) \hat{Q}'(X;t)$$

Conjecture (Novelli-T., in preparation)

The coefficient $c_G^{\mu}(t)$ is given by an explicit statistic $\mathrm{st}_G(\pi)$ on set partitions of type μ which are compatible with G, i.e. such that the extremities of an edge are not in the same block:

$$c_G^{\mu}(t) = \sum_{\pi \in \Pi_{\mu}} t^{\mathrm{st}_G(\pi)}$$

For the graph $G =$ 

$$LLT_G = \hat{Q}'_{11111} + (t^3 + 2t^2 + 2t)\hat{Q}'_{2111} + (t^3 + 2t^2 + t)\hat{Q}'_{221}$$

$\{\{1\},\{2,4\},\{3,5\}\}$	1
$\{\{1,4\},\{2\},\{3,5\}\}$	2
$\{\{1,4\},\{2,5\},\{3\}\}$	3
$\{\{1,5\},\{2,4\},\{3\}\}$	2
$\{\{1\},\{2\},\{3,5\},\{4\}\}$	1
$\{\{1\},\{2,5\},\{3\},\{4\}\}$	2
$\{\{1\},\{2,4\},\{3\},\{5\}\}$	1
$\{\{1,5\},\{2\},\{3\},\{4\}\}$	3
$\{\{1,4\},\{2\},\{3\},\{5\}\}$	2
$\{\{1\},\{2\},\{3\},\{4\},\{5\}\}$	0

Thanks to the Haglund-Wilson formula, this would provide an explicit expression of Macdonald polynomials in terms of Hall-Littlewood functions.

Conclusion

Bon anniversaire Bernard !