Attainable numbers and the Lagrange spectrum

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For any real number α the Lagrange constant $\mu(\alpha)$ is defined as follows

$$\mu^{-1}(\alpha) = \liminf_{p\in\mathbb{Z}, q\in\mathbb{N}} |q(q\alpha - p)|.$$

The set of all values taken by $\mu(\alpha)$ as α varies is called the *Lagrange spectrum* $\mathbb{L}$. Irrational α is called *attainable* if the inequality

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{1}{\mu(\alpha)q^2}$$

holds for infinitely many relatively prime integers p and q. We call the Lagrange spectrum element λ *admissible* if there exists an irrational number α such that $\mu(\alpha) = \lambda$. Malyshev in survey paper [M] claimed that every element of $\mathbb{L}$ is admissible. However, in the paper [G] the counterexample was constructed.

Theorem 1 *The quadratic irrationality* $\lambda_0 = [3;3,3,2,1,\overline{1,2}] + [0;2,1,\overline{1,2}]$ *belongs to* $\mathbb{L}$*, but is not an admissible number.*
The Lagrange spectrum is closed set. The complement of $\mathbb{L}$ is a countable union of *maximal gaps* of the spectrum. However, almost all points of $\mathbb{L}$ are admissible.

Theorem 2 *If* $\lambda \in \mathbb{L}$ *is not a left endpoint of some maximal gap in the Lagrange spectrum then* λ *is an admissible number.*
All left endpoints of maximal gaps of the Lagrange spectrum are also described in [G].

Theorem 3 *If* (a, b) *is a maximal gap in* $\mathbb{L}$ *then* a *can be represented by a sum of two quadratic irrationalities.* The author recently established the sufficient and necessary criteria of admissibility.

Theorem 4 *A Lagrange spectrum left endpoint* a *is admissible if and only if there exists a quadratic irrationality* α *such that* $\mu(\alpha) = a$.

[M] A.V. Malyshev, *Markov and Lagrange spectra (survey of the literature)*, p. 770, translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V.A. Steklova AN SSSR, Vol 76 (1977), 39–85.

[G] Gayfulin D. *Attainable numbers and the Lagrange spectrum*, accepted by Acta Arithmetica, to be published in 2017,
preprint at arXiv:https://arxiv.org/abs/1606.01600 (2016), 14 p.