Lattices and Diophantine exponents

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Let Λ be a lattice of rank d in $\mathbb{R}^d$. For each $\boldsymbol{v} = (v_1 \ldots v_d)$ let us denote $\Pi(\boldsymbol{v}) = |v_1 \ldots v_d|$. If Λ is algebraic, then $\Pi(\boldsymbol{v})$ is bounded away from zero at nonzero lattice points. The same holds if Λ is the image of an algebraic lattice under the action of a non-degenerate diagonal matrix. It is an open question whether other such lattices exist for $d \geqslant 3$. A negative answer to this question is known to imply the Littlewood conjecture.

We turn our attention to a more general situation, when $\Pi(\boldsymbol{v})$ can attain however small values. In this case it is reasonable to talk about the rate of tending $\Pi(\boldsymbol{v})$ to zero over a sequence of lattice points, which leads to the notion of a Diophantine exponent of Λ defined as

$$\omega(\Lambda) = \sup\left\{\gamma \in \mathbb{R} \;\middle|\; \Pi(\boldsymbol{v}) < |\boldsymbol{v}|^{-\gamma} \text{ admits } \infty \text{ solutions in } \boldsymbol{v} \in \Lambda\right\}.$$

In our talk we discuss all that is currently known about Diophantine exponents of lattices and pay special attention to the spectrum of this quantity.