A formula for the height of hypersurfaces in toric varieties with respect to toric line bundles equipped with semipositive toric metrics

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In [BPS], the Arakelov geometry of toric varieties has been written in terms of convex geometry. We deduce from the results therein a formula for the height of a hypersurface Y in a toric variety with respect to the choice of toric line bundles equipped with semipositive toric metrics $\overline{L}_0, \ldots, \overline{L}_{n-1}$.

More precisely, on a base adelic field $(K, (|\cdot|_v, n_v)_{v\in\mathfrak{M}_K})$, such height can be written as

$$h_{\overline{L}_0,\ldots,\overline{L}_{n-1}}(Y) = \sum_{v\in\mathfrak{M}_K} n_v \; \mathrm{MI}_M(\vartheta_{0,v}, \ldots, \vartheta_{n-1,v}, \rho_v^\vee),$$

where $\{\vartheta_{i,v}\}$ are concave functions associated to the choice of the metrized line bundles, $\rho_v^\vee$ is the Legendre-Fenchel dual of a suitable v-adic Ronkin function of Y and MI_M denotes the mixed integral of concave functions on a real vector space.

A well-known result by V. Maillot for the canonical height of hypersurfaces in smooth projective toric varieties follows easily from our general approach.

[BPS] J. I. Burgos Gil, P. Philippon, and M. Sombra, *Arithmetic geometry of toric varieties. Metrics, measures and heights*, Astérisque 360, Société Mathématique de France, 2014.