Totally Positive Quadratic Integers from the Numeration Point of View

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For a real quadratic field $K = \mathbb{Q}(\sqrt{D})$ with squarefree D we denote the semigroup of totally positive integers $\mathcal{O}_K^+ = \{\, x \in \mathcal{O}_K : x > 0 \text{ and } x' > 0 \,\}$. We say that $x \in \mathcal{O}_K^+$ is *indecomposable* if it cannot be written as $x = y + z$ with $y, z \in \mathcal{O}_K^+$. All indecomposable elements are easily obtained from the continued fraction coefficients of the integral generator $\sqrt{D}$ in case $D \equiv 2, 3 \pmod 4$ or $\frac{1+\sqrt{D}}{2}$ in case $D \equiv 1 \pmod 4$.

Knowledge about the indecomposable elements is crucial for the study of the universal quadratic forms over K. We investigate three arithmetic aspects of $\mathcal{O}_K^+$:

1. First, we show which elements of $\mathcal{O}_K^+$ are uniquely decomposable into indecomposables and how many of them are there modulo totally positive units in $\mathcal{O}_K$.

2. Second, we give a presentation of $\mathcal{O}_K^+$ as an aditive semigroup; the generators are the indecomposable elements and the relations are of the form $c\alpha = \beta + \gamma$ for certain indecomposables α, β, γ and integers $c \geq 2$. Note that $\mathcal{O}_K^+$ is not finitely generated in algebraic number fields other than $K = \mathbb{Q}$.

3. We then show that the indecomposable elements form a base of an integer numeration system over $\mathcal{O}_K^+$, i.e., that for each $K = \mathbb{Q}(\sqrt{D})$ there exists m such that every $x \in \mathcal{O}_K^+$ is expressible as a finite sum

$$x = \sum_{\alpha \text{ indecomposable}} x_\alpha \alpha \quad \text{with} \quad x_\alpha \in \{0, 1, \dots, m\}.$$

All three results are constructive in the sense that only the aforementioned continued fraction has to be computed in order to obtain the explicit formulas.

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