On the p-adic valuation of Stirling numbers of the first kind

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Given integers $n \geq k \geq 1$, let $s(n,k)$ be the Stirling number of the first kind, i.e., the number of permutations of $\{1,\ldots,n\}$ with exactly k disjoint cycles, and define $H(n,k) := \sum 1/(i_1 \cdots i_k)$, where the sum is extended over all integers $1 \leq i_1 < \cdots < i_k \leq n$. These quantities are related by the identity $H(n,k) = s(n+1,k+1)/n!$.

Several researchers [1, 2, 3, 4, 5, 7] have studied their p-adic valuations. In particular, it is an open conjecture that, for each prime p and positive integer k, there exist only finitely many n such that p divides $H(n,k)$. In this talk we shall illustrate the following result:

Theorem [L. and Sanna [6]]. *For each prime p, integer $k \geq 2$, and $x \geq (k-1)p$, there exists a constant $c = c(p,k) > 0$ such that*

$$\nu_p(H(n,k)) < -c\log(n)$$

for all positive integers $n \in [(k-1)p, x]$ whose base p representations start with the base p representation of $k-1$, but at most $3x^{0.835}$ exceptions.

We also provide a description of $\nu_2(H(n,2))$ in terms of an infinite binary sequence.

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