Binomial Polynomials mimicking Riemann's Zeta Function

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The (generalised) Mellin transforms of certain Chebyshev and Gegenbauer functions based upon the Chebyshev and Gegenbauer polynomials, have polynomial factors $p_n(s)$, whose zeros lie all on the 'critical line' $\Re s = 1/2$ or on the real axis (called critical polynomials). The transforms are identified in terms of combinatorial sums related to H. W. Gould's S:4/3, S:4/2 and S:3/1 binomial coefficient forms. Their 'critical polynomial' factors are then identified as variants of the S:4/1 form, and more compactly in terms of certain ${}_3F_2(1)$ hypergeometric functions. Furthermore, we extend these results to a 1-parameter family of polynomials with zeros only on the critical line. These polynomials possess the functional equation $p_n(s;\beta) = \pm p_n(1-s;\beta)$, similar to that for the Riemann xi function.

It is shown that via manipulation of the binomial factors, these 'critical polynomials' can be simplified to an S:3/2 form, which after normalisation yields the rational function $q_n(s)$. The denominator of the rational form has singularities on the negative real axis, and so $q_n(s)$ has the same 'critical zeros' as the 'critical polynomial' $p_n(s)$. Moreover as $s \to \infty$ along the positive real axis, $q_n(s) \to 1$ from below, mimicking $1/\zeta(s)$ on the positive real line.

In the case of the Chebyshev parameters we deduce the simpler S:2/1 binomial form, and with $\mathcal{C}_n$ the nth Catalan number, s an integer, we show that polynomials $4\mathcal{C}_{n-1}p_{2n}(s)$ and $\mathcal{C}_n p_{2n+1}(s)$ yield integers with only odd prime factors.