On the Density of Sets Avoiding Parallelohedron Distance 1

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In this joint work with C. Bachoc, T. Bellitto and A. Pêcher, we study the density of sets avoiding distance 1 in $\mathbb{R}^n$.

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. We consider the so-called unit distance graph G associated with $\|\cdot\|$: the vertices of G are the points of $\mathbb{R}^n$, and the edges connect the pairs $\{x, y\}$ satisfying $\|x-y\| = 1$. We define $m_1(\mathbb{R}^n, \|\cdot\|)$ as the supremum of the densities achieved by independent sets of G. The number m_1 was introduced by Larman and Rogers (1972) as a tool to study the measurable chromatic number $\chi_m(\mathbb{R}^n)$ of $\mathbb{R}^n$ for the Euclidean norm.

The best known estimates for $\chi_m(\mathbb{R}^n)$ and $m_1(\mathbb{R}^n, \|\cdot\|)$ present relations with Euclidean lattices, in particular with the sphere packing problem: for instance the best known lower bound on $m_1(\mathbb{R}^2, \|\cdot\|_2)$ has been obtained by using the optimal sphere packing in the plane, given by the hexagonal lattice.

The determination of $m_1(\mathbb{R}^n, \|\cdot\|)$ for the Euclidean norm is an open question for all dimensions $n \geq 2$. We study this problem for norms whose unit ball is a convex polytope that tiles space by translation. In that case, C. Bachoc and S. Robins conjectured that $m_1(\mathbb{R}^n, \|\cdot\|) = \frac{1}{2^n}$. Voronoi conjectured that the polytopes that tile space by translation are, up to affine transformations, Voronoi regions of lattices. This conjecture was proved by Delone for dimensions $n \leq 4$.

We prove Bachoc and Robins conjecture for $n = 2$ and for some families of Voronoi regions of lattices in higher dimensions. To do so, we turn the problem into a discrete problem, and we obtain our results by solving a packing problem in lattices.