Local-global divisibility in abelian varieties

Laura Paladino
Max Planck Institute for Mathematics - Bonn

Let k be a number field and let $\mathcal{A}$ be a commutative algebraic group defined over k. We denote by M_k the set of places $v \in k$ and by k_v the completion of k at the valuation v. We consider the following question.

Local-Global Divisibility Problem. *Let $P \in \mathcal{A}(k)$. Suppose that for all but finitely many $v \in M_k$, there exists $D_v \in \mathcal{A}(k_v)$ such that $P = qD_v$, where q is a positive integer. Is it possible to conclude that there exists $D \in \mathcal{A}(k)$ such that $P = qD$?*

The answer to that problem is linked to the behaviour of a cohomological group, whose definition is similar to the one of the Tate-Shafarevich group $Ш(k, \mathcal{A}[q])$, where $\mathcal{A}[q]$ denotes the q-torsion subgroup of $\mathcal{A}$.

We show a new result recently obtained for abelian varieties of dimension $g = 2^h$, $h \geq 0$, implying an affirmative answer to the local-global divisibility by a prime p under certain conditions. As a consequence, in the case when $\mathcal{A}$ satisfies those conditions, we have the triviality of $Ш(k, \mathcal{A}[p])$. If $\mathcal{A}$ is principally polarized, then the vanishing of $Ш(k, \mathcal{A}[p])$ implies a local-global principle for divisibility by p for the elements of $H^r(k, \mathcal{A})$, for all $r \geq 0$.