Parametric geometry of numbers in function fields

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In this joint work with Michel Waldschmidt, we transpose the parametric geometry of numbers, recently created by Schmidt and Summerer, to fields of rational functions in one variable and analyze, in that context, the problem of simultaneous approximation to exponential functions.

Consider the field $F((1/T))$ of Laurent series in one variable over an arbitrary field F, equipped with the absolute value $|f| = e^{\deg(f)}$. For each $\mathbf{u} = (u_1, \dots, u_n) \in F((1/T))^n$ with $\max\{|u_1|, \dots, |u_n|\} = 1$, each $i = 1, \dots, n$ and each $q \geq 0$, let $L_{\mathbf{u},i}(q)$ denote the minimum of all $t \geq 0$ for which the inequalities

$$\max\{|x_1|, \dots, |x_n|\} \leq e^t \quad \text{and} \quad |u_1x_1 + \cdots + u_nx_n| \leq e^{t-q}$$

admit at least i solutions $(x_1, \dots, x_n)$ in $F[T]^n$ which are linearly independent over $F(T)$.

1) We characterize all maps

$$\begin{aligned} \mathbf{L}_{\mathbf{u}} \colon [0,\infty) &\longrightarrow \mathbb{R}^n \\ q &\longmapsto (L_{\mathbf{u},1}(q), \dots, L_{\mathbf{u},n}(q)). \end{aligned}$$

2) Among them, there is a simplest one $\mathbf{P} = (P_1, \dots, P_n)$ satisfying $P_n(q) - P_1(q) \leq 1$ for each $q \geq 0$. We show that when F has characteristic zero and $\omega_1, \dots, \omega_n$ are distinct elements of F, the point $\mathbf{u} = (e^{\omega_1/T}, \dots, e^{\omega_n/T})$ has $\mathbf{L}_{\mathbf{u}} = \mathbf{P}$. The same applies to all perfect n-tuples of series in the sense of Mahler-Jager.

3) In 1964, A. Baker showed that, the n-tuple $\left(e^{\omega_1/T}, \dots, e^{\omega_n/T}\right)$ provides a counterexample to the analogue in $\mathbb{C}((1/T))$ of a conjecture of Littlewood. We also generalize this result to several places of $\mathbb{C}(T)$.