On the G.C.D. of n and the nth term of a linear recurrence

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Let $(u_n)_{n\geq 1}$ be a nondegenerate linear recurrence over the integers of order $k \geq 2$, and define $g_u(n) := \gcd(n, u_n)$ for all positive integers n. Several researchers [1, 2, 4, 6, 7] have studied the set of fixed points of g_u, that is, the set of positive integers n dividing u_n. In this talk we shall illustrate the following two results on the preimage and image of g_u.

Theorem [S. [5]]. *The set $g_u^{-1}(1)$, that is, the set of positive integer n such that u_n and n are relatively prime, has an asymptotic density. Moreover, this density is positive unless $(u_n/n)_{n\geq 1}$ is a linear recurrence.*

Theorem [Leonetti and S. [3]]. *If $(u_n)_{n\geq 1}$ is the sequence of Fibonacci numbers, then the set $\mathcal{A} := g_u(\mathbf{N})$ has asymptotic density zero and*

$$\#\mathcal{A}(x) \gg \frac{x}{\log x}$$

for all $x \geq 2$.

[1] J. J. Alba González, F. Luca, C. Pomerance, and I. E. Shparlinski, *On numbers n dividing the nth term of a linear recurrence*, Proc. Edinb. Math. Soc. (2) **55** (2012), no. 2, 271–289.

[2] R. André-Jeannin, *Divisibility of generalized Fibonacci and Lucas numbers by their subscripts*, Fibonacci Quart. **29** (1991), no. 4, 364–366.

[3] P. Leonetti and C. Sanna, *On the greatest common divisor of n and the nth Fibonacci number*, Rocky Mountain J. Math. (accepted).

[4] F. Luca and E. Tron, *The distribution of self-Fibonacci divisors*, Advances in the theory of numbers, Fields Inst. Commun., vol. 77, Fields Inst. Res. Math. Sci., Toronto, ON, 2015, pp. 149–158.

[5] C. Sanna, *On numbers n relatively prime to the nth term of a linear recurrence*,
`https://arxiv.org/abs/1703.10478`.

[6] C. Sanna, *On numbers n dividing the nth term of a Lucas sequence*, Int. J. Number Theory **13** (2017), no. 3, 725–734.

[7] L. Somer, *Divisibility of terms in Lucas sequences by their subscripts*, Applications of Fibonacci numbers, Vol. 5 (St. Andrews, 1992), Kluwer Acad. Publ., Dordrecht, 1993, pp. 515–525.