An algebraic approach to the Brauer-Siegel ratio for abelian varieties over function fields

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Let K be the function field of a curve $\mathcal{C}$ over a finite field $\mathbf{F}_q$, and let A be an abelian variety over K. Analogy with the classical Brauer-Siegel theorem led Hindry to consider the ratio

$$BS(A) := \frac{\log\left(|Ш(A)|\mathrm{Reg}(A)\right)}{\log H(A)}$$

where $|Ш(A)|$ is the order of the Tate-Shafarevich group of A, $\mathrm{Reg}(A)$ is the Néron-Tate regulator of A, and $H(A)$ is the exponential differential height of A.

If A_n, $n \geq 1$ is a sequence of abelian varieties of fixed dimension over K with $H(A_n) \to \infty$, the classical analogy would suggest that

$$\lim_{n\to\infty} BS(A_n) = 1.$$

Hindry, Pacheco, and Griffon have used analytic techniques to give several example of families $\{A_n\}$ for which the limit above exists and is equal to 1. (They also gave evidence for the conjecture that a limit of zero is possible.) More specifically, they use the formula of Birch and Swinnerton-Dyer (in cases where it is known to hold) and an estimation of the leading Taylor coefficient of the L-function of A_n over K to estimate the size of $|Ш(A)|\mathrm{Reg}(A)$.

In this talk, I will explain an algebraic technique for estimating $|Ш(A)|\mathrm{Reg}(A)$ directly which recovers the limit above in all cases considered by Hindry, Pacheco, and Griffon. One technical novelty in our approach is that we use properties of the function

$$m \mapsto |Ш(A/\mathbf{F}_{q^m}(\mathcal{C}))|$$

to estimate the size of $|Ш(A)|\mathrm{Reg}(A)$ over $K = \mathbf{F}_q(\mathcal{C})$, i.e., over the original ground field.