Dynamical Zeta Function of the β-shift for β a Totally Real Algebraic Number. The Bogomolov Property

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For $\beta > 1$ being an algebraic integer close to one and tending to one, the Mahler measure $\mathrm{M}(\beta)$ of β, the Weil height $h(\beta)$ and the dynamical zeta function $\zeta_\beta(z)$ of the β-shift are intimately correlated. In particular when all the conjugates of β are real.

Schinzel (1973) obtained the lower bound $\frac{1}{2}\log(1+\sqrt{5}/2) = 0.24\ldots$ for $h(\beta)$, and any totally real algebraic integer $\beta, \neq 0, \neq \pm 1$, optimally. We show that the problem of minoration of the height $h(\beta)$, $\beta \in \mathcal{O}_{\overline{\mathbb{Q}}}$, is related to the problem of Lehmer for Salem numbers α with the Mahler $\mathrm{M}(\alpha)$. Results of Langevin, C. Smyth and V. Flammang will be recalled. The extension from algebraic integers to algebraic numbers require other techniques.

Bombieri and Zannier (2001) have introduced the Bogomolov property for a field $F \subset \overline{\mathbb{Q}}$, by analogy with the Conjecture of Bogomolov: F is said to have the Bogomolov property relatively to h if and only if $h(\alpha) = 0$ or admits a lower bound > 0 for all $\alpha \in F$. Amoroso et Zannier (2000) have proved it for K^{ab}, where K is a number field, Bombieri and Zannier (2001) for totally p-adic number fields, Habegger (2011) for $\mathbb{Q}(E_{tors})$, $E/\mathbb{Q}$ an elliptic curve. Fili et Miner (2016), by using theorems of limit equidistribution of Favre and Rivera-Letelier, have proved $\liminf h(\alpha) \geq 0.12\ldots$ for α belonging to the field of totally real algebraic numbers $\mathbb{Q}^{tr}$. More recently, by other techniques, Pottmeyer (2016) obtained $\liminf h(\alpha) \geq \frac{7}{4\pi^2}\zeta(3)$.

In this work we obtain new properties of the dynamical zeta function $\zeta_\beta(z)$ of the β-shift, which imply in particular the Bogomolov property for $\mathbb{Q}^{tr}$, with an explicit minoration of the height.