Finite Spectral Representations of the Partition Function and Related Mappings

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A partition of a positive integer n is an additive decomposition of n into a sum where the summands are positive integers as well and the order of these summands does not matter. The partition function $p(n)$ counts the number of all partitions of n, and it is a traditional problem of constructive number theory to find an explicit *finite* representation of $p(n)$ (rather than infinite series expansions such as Rademacher's formula). Bruinier and Ono [2013] established the first finite representation of $p(n)$ as a finite sum of algebraic numbers being singular moduli for a certain weak Maass form. It is the purpose of this talk to establish several alternatives which exhibit a spectral nature of the partition function and related arithmetical functions. More precisely, there exists a sparse upper Hessenberg matrix H with the property that the partition function, certain divisor functions, Ramanujan's τ-function as well as their respective summatory functions admit finite representations in terms of powers or eigenvalues of H. For example, the partition function $p(n)$ and the divisor function $\sigma_1(n)$ are expressible in terms of the eigenvalues λ_j of H and certain coefficients α_j according to

$$p(n) = \sum_j \alpha_j \cdot \lambda_j^n \quad \text{and} \quad \sigma_1(n) = \sum_j \lambda_j^n,$$

and Ramanujan's τ-function has the matrix representation

$$\tau(n) = n^4 \cdot \mathrm{trace}(H^n) - 24 \sum_{k=1}^{n-1} (35k^4 - 60n^2k^2 + 13n^4) \cdot \mathrm{trace}(H^k) \cdot \mathrm{trace}(H^{n-k}).$$

In particular, these functions can be regarded as superpositions of sinusoids which explains their oscillating behaviour.

References

J.H. Bruinier and K. Ono: *Algebraic formulas for the coefficients of half-integral weight harmonic weak Maass forms,* Adv. Math. **246** (2013), 198-219

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