Effective results for linear Equations in Members of two Recurrence Sequences

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Let $(U_n)_{n=0}^{\infty}$ and $(V_m)_{m=0}^{\infty}$ be simple, non-degenerate, recurrence sequences with dominant roots α and β respectively. Moreover let us assume that $(U_n)_{n=0}^{\infty}$ and $(V_m)_{m=0}^{\infty}$ are defined over the integers. This talk is devoted to the study of the linear equation

$$a_1 U_{n_1} + \cdots + a_k U_{n_k} = b_1 V_{m_1} + \cdots + b_\ell V_{m_\ell},$$

where $a_1, \ldots, a_k$ and $b_1, \ldots, b_\ell$ are fixed non-zero integers. Under some mild technical restrictions we show that there exist only finitely many effectively computable solutions.

We apply this result to the following situation. Let $(G_n)_{n=0}^{\infty}$ be a linear recurrence of order d, given by

$$G_{n+d} = c_0 G_{n+d-1} + \cdots + c_{d-1} G_n,$$

where $d \geq 1$, $G_0 = 1$, $G_k = c_0 G_{k-1} + \cdots + c_{k-1} G_0 + 1$ for $k < d$ and c_i are non-negative integers for $i = 0, \ldots, d-1$. Then G can be used as the basis of a numeration system. Suppose we are given two such numeration systems with bases multiplicatively independent bases G and H and let us denote by $H_G(n)$ and $H_H(n)$ the Hamming weight of the digit expansion of n in bases G and H respectively. Then we show that there exists an effectively computable constant C such that for every $m \geq 2$ the inequality $H_G(n) + H_H(n) \leq m$ implies that

$$\log n < (Cm \log m)^{m-1}.$$