Compact quantum hypergroups

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Outline

A hypergroup structure on a C^*-algebra

$(A, \cdot, 1, *)$ is a separable unital C^*-algebra, $A \otimes A$ denotes the injective C^*-tensor square of A.

Definition

$(A, \delta, \epsilon, \star)$ is a hypergroup structure on the C^*-algebra $(A, \cdot, 1, *)$ if:

(HS$_1$) $(A, \delta, \epsilon, \star)$ is a $\star$-coalgebra with a counit ϵ, that is, $\delta : A \to A \otimes A$ and $\epsilon : A \to \mathbb{C}$, $\star : A \to A$, and

$$(\delta \otimes id) \circ \delta = (id \otimes \delta) \circ \delta,$$
$$(\epsilon \otimes id) \circ \delta = (id \otimes \epsilon) \circ \delta = id,$$
$$\delta \circ \star = \Pi \circ (\star \otimes \star) \circ \delta,$$
$$\star \circ \star = id,$$

where $\Pi : A \otimes A \to A \otimes A$ is the flip, $\Pi(a_1 \otimes a_2) = a_2 \otimes a_1$;

Definition (cont.)

(HS$_2$) $\delta : A \to A \otimes A$ is positive;

(HS$_3$) the following identities hold:

$$(a \cdot b)^\star = a^\star \cdot b^\star, \qquad \delta \circ * = (* \otimes *) \circ \delta,$$

$$\epsilon(a \cdot b) = \epsilon(a)\epsilon(b), \qquad \delta(1) = 1 \otimes 1,$$

$$\star \circ * = * \circ \star.$$

Examples

Example

Let $(A, \cdot, 1, *, \Delta, \epsilon, S)$ be a compact matrix pseudogroup with $\mathcal{A}$ being the involutive subalgebra generated by matrix elements of the fundamental corepresentation.[1]

Let

$$a^\star = f_{-1/2} \,.\, S(a)^* \,.\, f_{1/2}.$$

Then $(A, \Delta, \epsilon, \star)$ is a hypergroup structure on $(A, \cdot, 1, *)$.

[1] S. L. Woronowicz. Compact matrix pseudogroups. *Commun. Math. Phys.*, 111:613–665, 1987.

Example

Let $(A_1, \cdot, 1, *, \Delta_1, \epsilon_1, S_1)$ and $(A_2, \cdot, 1, *, \Delta_2, \epsilon_2, S_2)$ be compact matrix pseudogroups, and $\pi : A_1 \to A_2$ a Hopf C^*-algebra epimorphism. Let

$$\mathcal{A}_1/\mathcal{A}_2 = \{a \in \mathcal{A}_1 : (id \otimes \pi) \circ \Delta_1(a) = a \otimes 1\},$$

$$\mathcal{A}_2 \backslash \mathcal{A}_1 = \{a \in \mathcal{A}_1 : (\pi \otimes id) \circ \Delta_1(a) = 1 \otimes a\},$$

$$\mathcal{A} = \mathcal{A}_2 \backslash \mathcal{A}_1 \bigcap \mathcal{A}_1/\mathcal{A}_2.$$

Define $\delta : \mathcal{A} \to \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$, $\star : \mathcal{A} \to \mathcal{A}$, and $\epsilon \colon \mathcal{A} \to \mathbb{C}$ by

$$\delta = (id \otimes \nu_2 \circ \pi \otimes id) \circ (\Delta_1 \otimes id) \circ \Delta_1,$$

$$a^\star = f_{-1/2} \,.\, S_1(a)^* \,.\, f_{1/2}, \qquad \epsilon(a) = \epsilon_1(a), \qquad a \in \mathcal{A},$$

where ν_2 is the Haar measure on A_2.
If $A^{\mathrm{inv}} = \overline{\mathcal{A}}^{\|\cdot\|}$, then $(A^{\mathrm{inv}}, \delta, \epsilon, \star)$ is a hypergroup structure on $(A^{\mathrm{inv}}, \cdot, 1, *)$.

Unital Banach $*$-algebra srtucture on A°

A' denotes the set of all continuous linear functionals on the C^*-algebra A.

For $\xi, \eta \in A'$, $a \in A$, set

$$\begin{gathered}(\xi \cdot \eta)(a) = (\xi \otimes \eta)\delta(a) \\ \xi^+(a) = \overline{\xi(a^\star)},\end{gathered} \tag{1}$$

$$\|\xi\| = \sup_{\|a\|=1} |\xi(a)|. \tag{2}$$

Proposition

Let the product, involution and the norm on A' be given by (1) and (2). Then $(A', \cdot, \epsilon, {}^+)$ is a unital Banach $*$-algebra.

Haar Measure

Definition

A state $\nu \in A'$ is called a Haar measure on $(A, \delta, \epsilon, \star)$ if

$$(\nu \otimes id) \circ \delta(a) = (id \otimes \nu) \circ \delta(a) = \nu(a)1 \tag{3}$$

for all $a \in A$.

Definition

An element $a \in A$ is called positive definite if

$$\xi \cdot \xi^{+}(a) \geq 0 \tag{4}$$

for all $\xi \in A'$.

Existence of a Haar measure

Theorem

Let $(A, \delta, \epsilon, \star)$ be a hypergroup structure on a C^*-algebra A. Suppose that the linear space spanned by positive definite elements is dense in A. Then there exists a Haar measure ν, it is unique, and $\nu^+ = \nu$.

Compact quantum hypergroup

Definition

Let $(A, \delta, \epsilon, \star)$ be a hypergroup structure on a C^*-algebra $(A, \cdot, 1, *)$. Then $\mathcal{A} = (A, \cdot, 1, *, \delta, \epsilon, \star, \sigma_t)$ a compact quantum hypergroup if

(QH_1) δ is completely positive and the linear span of positive definite elements is dense in A;

(QH_2) σ_t, $t \in \mathbb{R}$, is a continuous one-parameter group of automorphisms of A such that

(a) there exist dense subslgebras $A_0 \subset A$ and $\tilde{A}_0 \subset A \otimes A$ such that the one-parameter groups σ_t and $\sigma_t \otimes id$, $id \otimes \sigma_t$ can be extended to complex one-parameter groups σ_z and $\sigma_z \otimes id$, $id \otimes \sigma_z$, $z \in \mathbb{C}$, of automorphisms of the algebras A_0 and $\tilde{A}_0$ respectively;

(b) A_0 is invariant with respect to $*$ and $\star$, and $\delta(A_0) \subset \tilde{A}_0$;

(c) the following relations hold for all $z \in \mathbb{C}$, $a \in A_0$:

$$\delta \circ \sigma_z = (\sigma_z \otimes \sigma_z) \circ \delta,$$
$$\nu(\sigma_z(a)) = \nu(a);$$

Compact quantum hypergroup

Definition (cont.)

(d) there exists $z_0 \in \mathbb{C}$ such that, for κ, defined by

$$\kappa = * \circ \sigma_{z_0} \circ \star,$$

the Haar measure ν satisfies the following strong invariance condition for all $a, b \in A_0$:

$$(id \otimes \nu)\big((\kappa \otimes id) \circ \delta(a) \cdot (1 \otimes b)\big) = (id \otimes \nu)\big((1 \otimes a) \cdot \delta(b)\big);$$

(QH_3) the Haar measure ν is faithful on A_0.

Example

The two previous examples of hypergroup structures give examples of compact quntum hypergroups.

Theorem (Kalyuzhnyi' 2001[2])

Let $\mathcal{A} = (A, \cdot, 1, *, \delta, \epsilon, \star, \sigma_t)$ *be a compact quantum hypergroup and B a unital* C^**-subalgebra of A and* $P\colon A \to B$ *a conditional expectation. Let* A_0 *and* $\tilde{A}_0$ *be dense subalgebras of A and* $A \otimes A$*, correspondingly, such that*

$$P(A_0) \subset A_0,$$
$$(P \otimes id)(\tilde{A}_0) \subset \tilde{A}_0, \qquad (id \otimes P)(\tilde{A}_0) \subset \tilde{A}_0.$$

Let P satisfy the following:

$$(P \otimes id) \circ \delta \circ P = (P \otimes P) \circ \delta = (id \otimes P) \circ \delta \circ P,$$
$$P \circ \star = \star \circ P, \qquad P \circ \sigma_z = \sigma_z \circ P, \qquad \nu \circ P = \nu.$$

Set

$$\tilde{\delta} = (P \otimes P) \circ \delta.$$

Then $(B, \cdot, 1, *, \tilde{\delta}, \epsilon, \star, \sigma_t)$ *is a compact quantum hypergroup.*

[2] A. A. Kalyuzhyi. Conditional expectations on quantum groups and new examples of quantum hypergroups. MFAT, 2001, **7**, No.4, 49–68.

A realization theorem

Theorem

Let $\mathcal{A}$ be a commutative compact quantum hypergroup. Let Q denote the spectrum of the commutative C^*-algebra, e stand for ϵ. Then $(Q, *, e, \delta, \nu)$ is a compact hypergroup.

Convolution operators

Let $\mathcal{A}$ be a compact quantum hypergroup,

$$\langle a, b \rangle = \nu(b^* a),$$

and H_ν the completion of A to a Hilbert space w.r.t. $\| \cdot \|_\nu$.

Proposition

For $a \in A_0$, define $T_a : A_0 \to A_0$ and $\dagger : A_0 \to A_0$ by

$$T_a(x) = (id \otimes \nu)\big((1 \otimes a) \cdot \delta(x)\big), \qquad a^\dagger = \kappa(a)^*.$$

Then

(a) *for $a \in A_0$, $T_a : H_\nu \to H_\nu$ is a Hilbert-Schmidt type operator;*

(b) *for $x \in H_\nu$, $a \in A_0$,*

$$\|T_a(x)\| \leq \|a\| \, \|x\|_\nu, \qquad T_a(H_\nu) \subset A;$$

(c) $T_a^* = T_{a^\dagger}$.

(d) *the set $\{T_a(b) : a, b \in A_0\}$ is total in H_ν w.r.t. $\| \cdot \|_\nu$;*

Convolution operators (cont.)

Proposition

The set $R = \{T_a(b) : a, b \in A_0, a^\dagger = a\}$ is total in A with respect to the C^*-norm.

Proposition

Let $a \in A_0$ and $a^\dagger = a$. Let $y = T_a(x)$ for some $x \in H_\nu$ and

$$y = \sum_{i=1}^{\infty} \langle y, v^{\lambda_i} \rangle v^{\lambda_i}$$

be the Fourier expansion of y with respect to an orthonormal set of the eigen vectors v^{λ_i} of the self-adjoint compact operator T_a, where λ_i are corresponding eigen values, $\lambda_i \neq 0$. Then $v^\lambda \in A$ and the series converges in the C^*-norm.

Corepresentations of a coalgebra

Definition

Let A be a Banach space, and $\mathcal{A} = (A, \delta, \epsilon)$ a coalgebra. For a Banach space V and a continuous linear map $\iota : V \to A \otimes V$, where $A \otimes V$ is the Banach space completion of the algebraic tensor product w.r.t. the injective cross-norm, (V, ι) is a corepresentation of $\mathcal{A}$ if

$$(\delta \otimes id) \circ \iota = (id \otimes \iota) \circ \delta,$$

$$(\epsilon \otimes id) \circ \iota = id,$$

If $\dim V < \infty$, then (V, ι) is finite dimensional.
Two finite dimensional corepresentations (V_1, ι_1) and (V_2, ι_2) of a coalgebra $\mathcal{A}$ are equivalent if there is an invertible operator $F : V_1 \to V_2$ such that

$$\iota_2 \circ F = (id \otimes F) \circ \iota_1.$$

Corepresentations of a coalgebra (cont.)

Definition

For finite dimensional (V, ι) and a basis $\mathcal{E} = \{e_i\}_{i=1}^d$ of V,

$$\iota(e_i) = \sum_{j=1}^{d} t_{ij} \otimes e_j, \qquad t_{ij} \in A,$$

and t_{ij} are matrix elements of (V, ι) w.r.t. the basis $\mathcal{E}$.

Corepresentations of a compact quantum hypergroup

Definition

Let $\mathcal{A} = (A, \cdot, 1, *, \delta, \epsilon, \star, \sigma_t)$ be a compact quantum hypergroup. Then (V, ι) is a corepresentation of $\mathcal{A}$ if (V, ι) is a corepresentation of the coalgebra (A, δ, ϵ).

Properties of finite dimensional corepresentations

Proposition

Let (V, ι) be a finite dimensional corepresentation of a compact quantum hypergroup $\mathcal{A}$. Then the matrix elements t_{ij} in any basis of V are analytic w.r.t. the one-parameter group σ_t.

Theorem

Let (V^p, ι^p) and (V^q, ι^q) be finite dimensional irreducible corepresentations of a compact quantum hypergroup $\mathcal{A}$. Let t_{ij}^p and t_{kl}^q denote matrix elements of the corresponding corepresentations. Then

$$\nu(t_{ij}^p \kappa(t_{kl}^q)) = 0 \tag{5}$$

if either the corepresentations are not equivalent or $i \neq l$.

Coinvolutive corepresentations

Let H be a Hilbert space with an inner product $(\cdot,\cdot)$ and (H,ι) a corepresentation of a compact quantum hypergroup $\mathcal{A}$.

Definition

A finite dimensional corepresentation (H,ι) of a compact quantum hypergroup $\mathcal{A}$ is called a †-corepresentation if

$$\sum_{i=1}^{d}(u,v_i)b_i = \sum_{i=1}^{d}(u_i,v)a_i^{\dagger}, \tag{6}$$

for all $u, v \in H$, where $\iota(u)=\sum_{i=1}^{d} a_i \otimes u_i$, $\iota(v)=\sum_{i=1}^{d} b_i \otimes v_i$, $a_i, b_i \in A$, $u_i, v_i \in H$, and $d = \dim H$.

Properties

Proposition

Let t_{ij}, $i, j = 1, \dots, d$, be matrix elements of a finite dimensional †-corepresentation (H, ι) with respect to an orthonormal in H. Then

$$t_{ij}^{\dagger} = t_{ji}.$$

Corollary

If the corepresentations (V^p, ι^p) and (V^q, ι^q) are irreducible †-corepresentations, then

$$\nu(t_{ij}^p t_{lk}^{q\,*}) = 0 \tag{7}$$

if either the corepresentations are not equivalent or $i \neq l$.

Properties (cont.)

Proposition

Let (H, ι) be a finite dimensional †-corepresentation of a compact quantum hypergroup $\mathcal{A}$. Then (H, ι) is a finite direct sum of irreducible finite dimensional †-corepresentations, i.e. $H = \oplus_{i=1}^{k} H_i$ and (H_i, ι_i) is an irreducible †-corepresentation with $\iota_i = \iota|_{H_i}$.

Main results

Theorem

Let V be a Banach space, (V, ι) be an irreducible corepresentation of a compact quantum hypergroup $\mathcal{A}$. Then V is finite dimensional.

Theorem

Let Q be the set of all finite dimensional irreducible nonequivalent †-corepresentations (V^q, ι^q), $q \in Q$, of a compact quantum hypergroup $\mathcal{A}$ and $\mathcal{B} = \{t_{ij}^q : q \in Q, \ i, j = 1, \ldots, d_q = \dim V^q\}$ be the set of all matrix elements of these corepresentations with respect to some bases. Then the linear span of the set $\mathcal{B}$ is dense in A with respect to the C^*-norm.

Corollary

Let $\mathcal{B}$ be defined as in the Theorem. Then the linear span of $\mathcal{B}$ is total in H_ν with respect to the L_2-norm.