Examples
Kac algebras and subfactors

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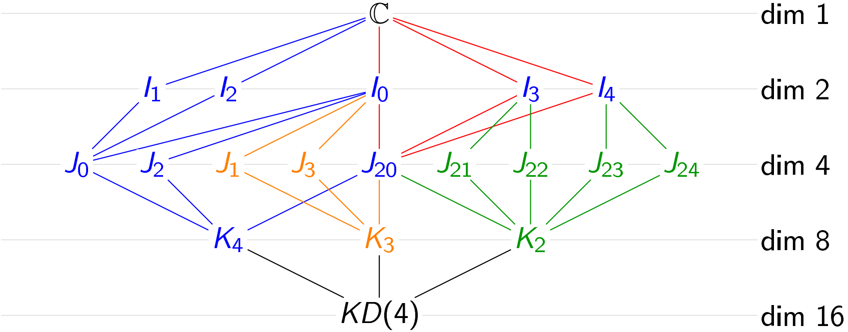
History

Kac algebras and subfactors

Intermediates factors and co-idalgebras

Exploration of Kac algebras $KD(n)$ (work with Nicolas M. Thiéry)



Kac algebras and subfactors

- **Kac algebras :** The theory of Kac algebras provides a unified framework for both group algebras and their duals.
  - G. Kac (1960) : *Ring groups*

Finite dimensional example

Kac, Paljutkin (1966) : KP de dim 8

  - Leonid worked with George Kac from 1968 until his sudden death in 1978.
  - M. Enock et J.-M. Schwartz : from 1970 to book in 1992
- Subfactors : A playground for Kac algebras
  - V. Jones *Index for subfactors (1983)*
  - D.Bisch (1992) : A note on intermediate subfactors.

Irreducible inclusion of depth 2, quantum groups

- A. Ocneanu 199..
  - T. Yamanouchi (1993) : Outer action of an fd Kac alg. on R.
  - *finite index* : Szymanski, Longo, (1994) MCD. (1996)
  - *general case* : Enock et Nest
- Sano (1996), MCD (1998) :

Tensorial product, bicrossed product and bicommutative square.

$$\begin{array}{ccccccc}
N^{\mathbb{A}} \cap N^{\mathbb{B}} & \subset & N^{\mathbb{B}} & & & & \\
\cap & & \cap & & & & \\
N^{\mathbb{A}} & \subset & N & \subset & N \rtimes \mathbb{A} & = & N^{\mathbb{B}} \rtimes (\mathbb{A} \rtimes_{\gamma_b} \mathbb{B}) \\
 & & \cap & & \cap & & \\
 & & N \rtimes \mathbb{B} & \subset & L & = & N \rtimes (\mathbb{A} \otimes_T \mathbb{B})
\end{array}$$

- M. Isuki, H. Kosaki : **Kac algebras** arising from composition of subfactors (2002).

$$G = H.K \quad and \quad R^H \subset R \rtimes K \quad \text{prof } 2$$

a lot of examples (dim ≤ 24) and classification

Some examples

- Y. Sekine : An example of finite dimensional Kac algebras of Kac-Paljutkin type (1996).

- **Twisting of group algebras**
  - Twisted coproduct
    - Enock, Vainerman (1996) : KP, $KD(3)$ (dim 12)
    - Vainerman, Nikshych (1998) : $KD(n)$ et $KQ(n)$ de dim $4n$
  - Twisted product :
    A. Masuoka : Cocycle deformations and Galois objects (2000).
    A_{4m}, B_{4m}

C^* quantum groupoids for reducible inclusions

- **Weak Kac Algebras , weak Hopf algebras**
  - Yamanouchi (1994)
  - Böhm, Nill et Szlachányi (1998)
  - Vainerman, Nikshych (1998)
- Finite index reducible inclusion of depth 2 and finite dimensional C^* **quantum groupoids**
  - D. Nikshych et L. Vainerman : A characterisation of depth 2 subfactors of II_1 factors (2000)

Examples

3 quantum groupoids for the 3 inclusions of depth 2 and index 4
a quantum groupoid of dimension 13 (see also Böhm,Szlachányi (1996))

- MCD (2005) : autodualité de la structure de C^*-groupoïde quantique des algèbres de Temperley-Lieb,
  action d'un C^*-groupoïde quantique sur R (D. Nikshych).

Type II_1 hyperfinite factor (Murray-von Neumann 1937)

The type II_1 hyperfinite factor is a von Neumann algebra, i.e. a weakly closed operator algebra ...

The type II_1 hyperfinite factor by Leonid (according to Sébastien's slide)
It is the von Neumann algebra of an ICC amenable group.

- $\mathbb{C}\Gamma = Vect\{\lambda_g | g \in \Gamma\}$ (Let Γ a countable group)
- $L(\Gamma)$ is the weak closure of $\mathbb{C}\Gamma$ acting on $l^2(\Gamma)$.
- $L(\Gamma)$ is a factor iff $\forall h \neq e \quad |\{ghg^{-1} | g \in \Gamma\}| = +\infty$
- $\tau(x) = \langle \delta_e, x\delta_e \rangle$ is a trace.
- ICC groups : S_∞, $\mathbb{F}_n$ $(n \geq 2)$, ...

- The type II_1 hyperfinite factor R is an infinite dimensional algebra reminiscent of $L^\infty(X,\mu)$ and $M_n(\mathbb{C})$
- **Properties of R :**
  - hyperfini : $R = \overline{\cup_{n>1} M_{2^n}(\mathbb{C})}$
  - facteur : $R' \cap R = Z(R) = \mathbb{C}$
  - type II_1 : finite normal faithful normalised trace $(tr(xy) = tr(yx))$ such that $tr(\mathcal{P}) = [0;1]$.
    For $(x, y \in R)$, $\langle x, y\rangle = tr(y^*x)$ is a scalar product.
- $L^2(R, tr)$ est l'**espace de Hilbert complété** de $(R, \langle ., .\rangle)$:

$$R \overset{\Lambda}{\hookrightarrow} L^2(R, tr)$$

- $L^\infty(X,\mu)$ acts on the left on $L^2(X,\mu)$ by multiplication :

$$M_f : \begin{array}{ccc} L^2(X,\mu) & \to & L^2(X,\mu) \\ g & \mapsto & fg \end{array}$$

- R acts on the left on $L^2(R, tr)$ by multiplication :

$$y\Lambda(x) = \Lambda(yx) \quad (x \in R, y \in R)$$

Inclusions of type II_1 factors (Jones 83)

Let $N_0 \subset N_1$ a finite index inclusion of type II_1 hyperfinite factors and tr the trace of N_1.

- Examples : $R^G \subset R$ or, for K Kac alg. $R^K \subset R$
- Jones projection of $N_0 \subset N_1$:
  orthogonal projection $f_1 : L^2(N_1, tr)$ on $L^2(N_0)$

$$N_0 = \{f_1\}' \cap N_1$$

  Ex : $f_1 = \sum_{g \in G} \frac{1}{|G|} u_g$
- Basic construction : factor $N_2 = \langle N_1, f_1 \rangle$ with the trace :
  $tr(f_1 x) = \tau tr(x) \qquad (x \in M_1)$
  with $\tau^{-1} = [N_1 : N_0]$ index of the subfactor.
- With the basic construction again,we get the **Jones tower** :

$$N_0 \subset N_1 \overset{f_1}{\subset} N_2 \overset{f_2}{\subset} N_3 \subset \cdots$$

Depth 2 irreducible subfactor (finite index)

- An invariant of the inclusion is the **derived tower**

$$N_0' \cap N_1 \subset N_0' \cap N_2 \overset{f_2}{\subset} N_0' \cap N_3 \dots$$

- $N_0 \subset N_1$ depth 2 if $N_0' \cap N_3 = \langle N_0' \cap N_2, f_2 \rangle$
- **group action :**

Jones Tower : $N_0 = R^G \subset N_1 = R \subset N_2 = R \rtimes G$
Relative commutants :
$N_0' \cap N_2 = \mathbb{C}[G]$ and $N_1' \cap N_3 = L^\infty(G)$ in $N_0' \cap N_3$.

- **general depth 2 subfactor**
  - irreducible : $N_0' \cap N_1 = \mathbb{C}$
  - $A := N_0' \cap N_2$ and $B := N_1' \cap N_3$ finite dim. C^*-algebras
  - duality : $\langle a, b \rangle = [N_1 : N_0]^2 tr(af_2f_1b) \quad \to \quad$ Kac algebras
  - A acts on N_1 :
    - $N_0 = N_1^A$, fixed points algebra under the action of A
    - $N_2 = N_1 \rtimes A$

Kac algebras, twisting group algebras (Vainerman 98)

Definition

$(K, \Delta, S, \varepsilon)$ is a finite dimensional C^*-Hopf-algebra or Kac algebra

- K is a finite dimensional C^*-algebra $(m, *, 1)$:

Example : Algebras of the dihedral group and quaternion group

$$D_{2n} = \langle a, b | a^{2n} = 1, b^2 = 1, ba = a^{-1}b \rangle \quad \rightarrow \quad KD(n)$$
$$Q_n = \langle a, b | a^{2n} = 1, b^2 = a^n, ba = a^{-1}b \rangle \quad \rightarrow \quad KQ(n)$$

$$K = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3 \oplus \mathbb{C}e_4 \oplus M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \ldots M_2(\mathbb{C})$$

- $n - 1$ factors $M_2(\mathbb{C})$
- $\dim(K) = 4n$
- with the normalised canonical trace :
  $tr(e_i) = 1/4n, \quad tr(e_{j,j}) = 1/2n, \quad tr(1) = 1$

Definition (continued) : K is a co-algebra

- Δ a coassociative coproduct on K :
  homomorphism from K to $K \otimes K$,
  $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$

Standard coproduct, twisting the standard coproduct with a 2-(pseudo) cocycle

- For $\mathbb{C}D_{2n}$: $\Delta_s(\lambda(g)) = \lambda(g) \otimes \lambda(g)$
- For $KD(n)$: $\Delta(x) = \Omega \Delta_s(x) \Omega^*$
  with Ω 2-(pseudo) cocycle (a unitary) of $H \otimes H$
  and $H = \{1, a^n, b, ba^n\}$.

- ε a counity : $\varepsilon(x) = \dim K \; tr(e_1 x) \quad (x \in K)$
  e_1 is Jones projection of the inclusion
- S an antipode, unique if Δ and ε are given.

$$m \circ (\mathrm{id} \otimes S) \circ \Delta = \varepsilon$$

Galois correspondence

Intermediates subfactors $\longleftrightarrow$ co-idalgebras

For an inclusion by group action $(N_0 = R^G \subset N_1 = R)$:

$$R^G \subset M \subset R \quad \Longleftrightarrow \quad M = R^H \quad \text{avec} \quad H < G$$

Inclusion of depth 2 (Vainerman, Nikshych 2000)

$$N_1 \subset M_1 \subset N_2 \overset{p}{\subset} M_3 = \langle N_2, p \rangle \subset N_3 = N_2 \rtimes K$$

- Jones projections of intermediate subfactors (D. Bisch 1992)
- $I(p) = N_1' \cap M_3 \subset K = N_1' \cap N_3$
- $M_3 = N_2 \rtimes I(p)$

Definition (co-idalgebra)

An involutive left coideal subalgebra I of K is a C^*-subalgebra I with unit such that $\Delta(I) \subset K \otimes I$

$I(K)$, the lattice of the co-idalgebras of K

- Isomorphism
  lattice of intermediates factors of $N_2 \subset N_2 \rtimes K$ / lattice $I(K)$
- $I \subset J \iff p_I$ dominates p_J (lattice of Jones projections)
- **Antiisomorphism of lattices** δ from $I(K)$ to $I(\widehat{K})$:

$$\delta(I) = S(I)' \cap \hat{K} \text{ in } \widehat{K} \rtimes K$$
$$\dim I \times \dim \delta(I) = \dim K$$

For groups

$$\begin{array}{ccc} I(\mathbb{C}[G]) & \to & I(L^\infty(G)) \\ \mathbb{C}[H] & \mapsto & \text{invariant functions modulo } H \end{array}$$

- Self-duality of K $\implies$ $I(K)$ is antiisomorphic to itsself.
- The Bratelli diagram of $I \subset K$ gives the principal graph of $R \subset R \rtimes \delta(I)$
  The principal graph of an inclusion is an invariant in relation with the derived tower.

How to find co-idalgebras ?

How to find subgroups in a group ?

Propositions (Vainerman, Nikshych 2000)

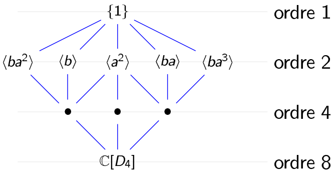
- *The restriction of* ε *to* I *is a positive linear form on* I*.*
- *There exists an unique projection* p_I *(called Jones projection of* I*) in the center of* I *such that :*

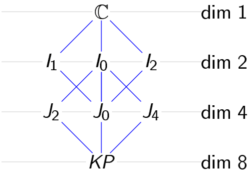
$$\forall x \in I \quad \varepsilon(x) = \dim I \; tr(p_I x) \quad et \quad Ip_I = \mathbb{C}p_I$$

- p_I *dominates* e_1*.*
- $\dim I = [M_3 : N_2] = tr(p_I)^{-1}$ *divides* $\dim K$*.*
- *The right legs of* $\Delta(p_I) = \sum_k S(x_k^*) \otimes x_k$ *span* I*.*
  Notation : $I = I(p_I)$
- $\Delta(p_I)(1 \otimes p_I) = p_I \otimes p_I$

A first example

KP is a deformation of $\mathbb{C}[D_4]$.





For bigger examples, I needed help !

Exploration of $KD(n)$ with Nicolas M. Thiéry and MuPAD-Combinat (now Sage-Combinat).
(to be published in Journal of Algebra and Its Applications)

Some general results for $KD(n)$

Proposition

- *3 co-idalgebras of dimension* $2n$: $K_i := I(e_1 + e_i)$
- $K_2 := I(e_1 + e_2)$ *is commutative ; it is isomorphic to* $L^\infty(D_n)$
- *Co-idalgebras of dimension dividing* $2n$ *are in* K_2, K_3 *or* K_4.

Proposition

- *If* $d|n$ *then* $KD(d) \hookrightarrow KD(n)$ *via* $a \mapsto a^{n/d}$ *(russian dolls)*
- *All co-idalgebras containing* H *arise this way.*

Corollary

Intrinsic group of $KD(n)$ *:*

- *if* n *odd, it is* $H \to J_0$
- *if* n *even, it is* $D_4 \to K_0$

Automorphisms, isomorphisms, self-duality

Theorem

$KD(2m)$ is isomorphic to $KQ(2m)$.

$$\phi : \begin{cases} a & \mapsto \frac{1}{4}(a - a^{-1})(a^m - a^{-m})(a^m - b) + a \\ b & \mapsto \frac{1}{2}(a^m + a^{-m})(b + \mathrm{i}) - \mathrm{i}a^m \end{cases}$$

Theorem

$KD(2m+1)$ and $KQ(2m+1)$ are self-dual.
Isomorphism for $KD(n)$:

$$a \mapsto n(2\widehat{e^{n-1}_{1,1}} + \widehat{e^{1}_{2,2}} - \widehat{e^{1}_{1,1}} - \widehat{e^{n-1}_{1,2}} - \widehat{e^{n-1}_{2,1}}), \qquad b \mapsto 4n\widehat{e_4}$$

Theorem

Automorphism group of $KD(n)$ and $KQ(n)$: $\mathbb{Z}^*_{2n} \rtimes \mathbb{Z}_2$

$$a \mapsto a^r, \qquad b \mapsto b, \qquad pour\ r \wedge 2n = 1$$

$$a \mapsto a - \frac{1}{2}(a - a^{-1})(1 + a^n), \qquad b \mapsto a^n b$$

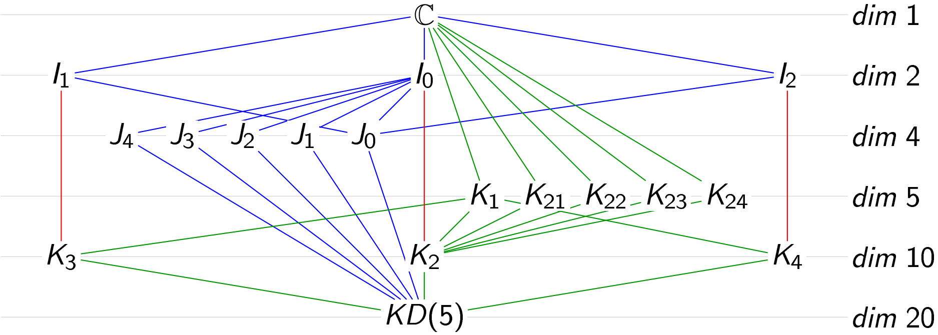
Algorithm for computing (all) Kac algebra isomorphisms

- $K := K(\mathbb{C}[G], H, \Omega)$: Kac algebra obtained by twisting some $\mathbb{C}[G]$ via a 2-pseudo cocycle Ω of $H < G$
- K' : any Kac algebra
- Problem : $K \approx K'$? Explicit isomorphism ϕ ?
- If yes : $K'' := K(K', \phi(H), \phi(\Omega^*)) \approx \mathbb{C}[G]$
- Algorithm :
  - Compute the intrinsic group H' of K'
  - Compute possible embeddings ρ of H into H'
  - Define $K'' := K(K', \rho(H), \rho(\Omega^*))$
  - Compute the intrinsic group G'' of K''
  - If $K'' = \mathbb{C}[G'']$, compute isomorphisms from G to G'' compatible with ρ.

The lattice of co-idalgebras of $KD(n)$ (n prime)

Theorem

For n prime, the lattice is like this of $KD(5)$:



The lattice of co-idalgebras of $KD(n)$ (n odd)

Propositions

- $J_0 = \mathbb{C}[H]$
- *Dim 2 : algebras of the 3 subgroups of H*
- *Dim* $2n$: K_2, K_3, K_4
- $K_2 = L^\infty(D_n)$
- K_3 *and* K_4 *isomorphics but not Kac subalgebras*
  $\Delta(e_1 + e_4) \implies$ *unités matricielles de* K_4
- *Co-idalgebras of dimension dividing n are in* K_2
- *Dim n : n co-idalgebras*
  (constant functions modulo subgroups of order 2 of D_n*)*
- *Dim 4 : n co-idalgebras*
- *For n prime, the list is complete.*

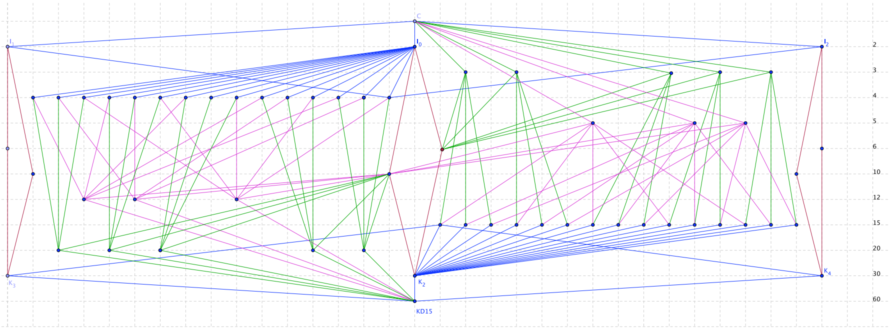
The lattice of co-idalgebras of $KD(n)$ (n odd)

More co-idalgebras in K_2.

And the co-idalgebras $I_2 \subset I \subset K_4$ of dim $2k|2n$? Idem in K_3.

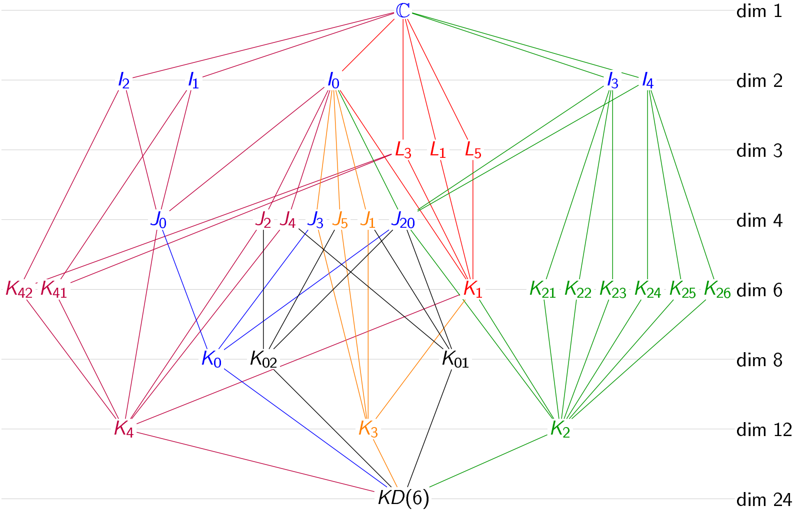
Conjecture

For n odd, the lattice is like that of $KD(15)$ *:*



true for $n \leq 51$. dim $KD(51) = 404$. A week for the computer !

Co-idalgebras lattice of $KD(6)$



$K_4 = KD(3) \quad K_3 = KQ(3) \quad K_2 = L^\infty(D_6) \quad K_0 = \mathbb{C}[D_4]$

The lattice of co-idalgebras of $KD(2m)$

Propositions

- $K_2 = L^\infty(D_n)$
- *Intrinsic group algebra* K_0 *is generated by* a^m *and* b *and isomorphic à* $KD(2)$*,*
- K_4*, generated by* a^2 *and* b*, is isomorphic to* $KD(m)$ *(russian dolls)*
- *Co-idalgebras de dim dividing* $2n$ *are in* K_2*,* K_3*,* K_4
- *For* m *odd,* K_3 *is isomorphic to* $KQ(m)$*.*
  For every m*,* K_3 *is the* **self-dual** *Kac algebra* B_{4m} *(A. Masuoka 2000). We show that it is not a twisted group algebra.*

- The family $KD(n)$ contains the others families KP, $A_{4n} = \widehat{KD(n)}$, B_{4n}, $KQ(n)$)
- It has a rich and various structure.
- Kac algebras and subfactors go together
- To apply theorems on examples gives life to theorems.
- The exploration of examples needs computer.

Thank you, Leonid !