Podleś spheres, quantum groupoids and special functions

Kenny De Commer

Université de Cergy-Pontoise

March 13 2012

(Joint project with E. Koelink)

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- Action $X \circlearrowleft G \;\Rightarrow\;$ groupoid $X \rtimes G$.

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- Morphisms: $x \xrightarrow{g} y$ if $y = xg$.
- Composition: $x \xrightarrow{g} y \xrightarrow{h} z$, $x \xrightarrow{gh} z$.

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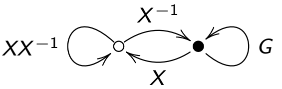
Enveloping groupoid:



Finite-dimensional weak Hopf C*-algebras

Definition (Böhm-Nill-Szlachányi)

Finite-dimensional weak Hopf C-algebra*:

- Finite-dimensional C*-algebra H,
- Coalgebra structure (H, Δ, ε),
- Δ is coassociative *-homomorphism *but not necessarily unital*.

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Existence of *source* and *target* algebras:

- Unital sub-C*-algebras H_s and H_t of H.
- $H_s \cong H_t^{\mathrm{op}}$.
- H_s and H_t commute.

Duality and representations

(H, Δ) is C^*-WHA $\Rightarrow (\widehat{H}, \widehat{\Delta})$ is C^*-WHA.

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(H,Δ) is C*-WHA $\Rightarrow$ $(\widehat{H},\widehat{\Delta})$ is C*-WHA.

$$\text{Unital } {}^*\text{-representation } \pi : \widehat{H} \to B(\mathscr{H})$$
$$\updownarrow$$
$$\text{Partial isometry corepresentation } \mathcal{G}_\pi \in H \otimes B(\mathscr{H}).$$

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$\widehat{H}_s \cong H_t,\ \widehat{H}_t \cong H_s \ \Rightarrow\ \mathscr{H}$ is H_t-bimodule.

Ehresmann-Schauenburg construction

Finite-dimensional C*-algebra A, finite-dimensional Hopf C*-algebra H.

Coaction $\alpha : A \to A \otimes H \Rightarrow$ C*-algebra $\mathcal{L} \subseteq A \otimes A^{\mathrm{op}}$:

$$\mathcal{L} = \{\sum_i a_i \otimes b_i^{\mathrm{o}} \mid \sum_i a_{i(0)} \otimes a_{i(1)} \otimes b_i^{\mathrm{o}} = \sum_i a_i \otimes S^{-1}(b_{i(1)}) \otimes b_{i(0)}^{\mathrm{o}}.\}$$

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Coaction *is free* (Galois condition): $\rightsquigarrow$ C*-WHA structure on L.

- Source/target algebras: $\mathcal{L}_s = Z$ and $\mathcal{L}_t = Z^{\mathrm{op}}$ where $Z = A^H$.
- Enveloping weak Hopf C*-algebra: $\mathcal{E} = \mathcal{L} \oplus A^{\mathrm{op}} \oplus A \oplus H$.
- Dual: $\widehat{\mathcal{E}} = \begin{pmatrix} \widehat{\mathcal{L}} & \widehat{A} \\ (\widehat{A})^* & \widehat{H} \end{pmatrix}$, with all entries coalgebras.

Quantization transformation groupoid

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However: with $A = P \rtimes K \;\rightsquigarrow\;$ dual right coaction $A \to A \otimes H$ is free.

$\Rightarrow$ C*-WHA structure on $\mathcal{L} \cong P^{\mathrm{op}} \rtimes K \ltimes P$ (Nikshych-Vainerman).

Source/target algebra: $\mathcal{L}_s \cong P$, $\mathcal{L}_r \cong P^{\mathrm{op}}$.

Morita base change

If H is C*-WHA with $H_s \underset{\text{Mor}}{\sim} B$

$\Rightarrow$ New C*-WHA $\widetilde{H}$ with $\widetilde{H}_s \cong B$.

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In particular, we can take B commutative (Hayashi).

Construction scheme

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Remarks:

- Both constructions special case of Ehresmann construction for free C^*-WHA actions.
- P. Schauenburg: setting of Hopf algebras/Hopf algebroids.
- J. Bichon, K. De Commer, M. Enock: setting of (locally) compact quantum groups/measured quantum groupoids.
- In particular, construction works for $K = \mathscr{L}^\infty(\mathbb{G})$ and $\mathscr{H}_i$ infinite-dimensional.

Actions of compact quantum groups on von Neumann algebras

Given $\mathbb{G}$ compact quantum group.
Goal: find actions of $\mathbb{G}$ on direct sums of type I-factors.
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Example: for each $0 < q < 1$ and $x \in \mathbb{R}$:

$$\mathbb{C}P^n_{q,x} \curvearrowleft SU_q^{n+1}, \quad \mathscr{L}^\infty(\mathbb{C}P^n_{q,x}) \cong \oplus_{i=1}^{n+1} B(l^2(\mathbb{N})).$$

Podleś spheres

Generators X, Y and Z, with $X^* = Y$ and $Z^* = Z$, and e.g.

$$\begin{aligned} XY &= (1 + q^{-x+1}Z)(1 - q^{x+1}Z) \\ YX &= (1 + q^{-x-1}Z)(1 - q^{x-1}Z). \end{aligned}$$

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Coaction γ_x by $\mathrm{Pol}(SU_q(2))$: $(X, q^{-1}Z - \frac{q^{-x}-q^{x}}{1+q^2}, Y)$ as spin 1.

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One has $\mathscr{L}^\infty(\mathbb{C}P^1_{q,x}) \cong B(l^2(\mathbb{N})) \oplus B(l^2(\mathbb{N}))$ by $\pi_- \oplus \pi_+$ where

$$\begin{aligned} \pi_\pm(Z)e_k &= \pm q^{2k\mp x+1}e_k \\ \pi_\pm(X)e_k &= \pm(1-q^{2k})^{1/2}(1+q^{2k\mp 2x})^{1/2}e_{k-1}. \end{aligned}$$

Structure of enveloping algebra

We have $\mathscr{L}^\infty(\mathbb{C}P^1_{q,x} \rtimes SU_q(2)) \cong \oplus_{j\in\mathbb{Z}} B(\widetilde{\mathscr{H}})$ with

$$\widetilde{\mathscr{H}} \cong l^2(\mathbb{N}) \otimes (l^2(\mathbb{N}) \oplus l^2(\mathbb{N})).$$

Then

$$E_x = E_{\gamma_x} = E_{11} \oplus E_{21} \oplus E_{12} \oplus E_{22}$$

with

$$\begin{aligned} E_{22} &\cong \mathscr{L}^\infty(\widehat{SU}_q(2)) \\ &\cong \oplus_{n\in\mathbb{N}_0} B(\mathbb{C}^n), \\ E_{12} &\cong \oplus_{j\in\mathbb{Z}} B(l^2(\mathbb{N}) \oplus l^2(\mathbb{N})). \end{aligned}$$

Structure of dual of enveloping algebra

Dual of enveloping quantum groupoid: $F_x = \widehat{E}_x$:

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \supseteq \begin{pmatrix} \mathbb{C}^2 \otimes \mathbb{C}^2 & 0 \\ 0 & \mathbb{C}. \end{pmatrix},$$

where

$$\begin{aligned} F_{22} &\cong \mathscr{L}^\infty(SU_q(2)) \\ &\cong B(l^2(\mathbb{N})) \otimes \mathscr{L}(\mathbb{Z}), \\ F_{12} &\cong \bigoplus_{\mu,\nu \in \{-,+\}} B(l^2(\mathbb{N}), \mathscr{K}^{(\mu\nu)}) \otimes \mathscr{L}(\mathbb{Z}). \end{aligned}$$

where

$$\mathscr{K}^+ = l^2(\mathbb{Z}), \quad \mathscr{K}^- = l^2(\mathbb{N}).$$

Corepresentations

Under this correspondence:

$$\text{The }{}^*\text{-representation } \pi^{(j)} : E_{12} \to B(l^2(\mathbb{N})_+^{(j)} \oplus l^2(\mathbb{N})_-^{(j)}),\ j \in \mathbb{Z}$$

$$\updownarrow$$

The coisometries

$$\mathcal{G}^{(j;\mu,\nu)} \in (B(l^2(\mathbb{N}), \mathscr{K}^{(\mu\nu)}) \otimes \mathscr{L}(\mathbb{Z})) \otimes B(l^2(\mathbb{N})_\nu, l^2(\mathbb{N})_\mu).$$

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$\rightsquigarrow$ matrix coefficients

$$\langle \mathcal{G}_{r,s}^{(j;\mu,\nu)}(e_m \otimes e_l), e_p \otimes e_k \rangle,$$

determined in terms of big q-Laguerre polynomials of degree m.

Associated quantum groupoid

We have $(F_{x,11}, \Delta) \cong (F_{x+k,11}, \Delta) \Leftrightarrow k \in \mathbb{Z}$.

$\Rightarrow$ circle-valued family of quantum groupoids with base of two points.

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Further: $(F_{0,11}/\mathbb{Z}_2, \Delta) \cong (\mathscr{L}^\infty(\widetilde{SU}_q(1,1)), \Delta)$ (Koelink-Kustermans).

The infinitesimal picture

Infinitesimal *-algebras (no x-dependence!):

$$\begin{array}{lcl} E_{22} & \leftrightsquigarrow & U_q(+,+) = U_q(su(2)), \\ E_{12} & \leftrightsquigarrow & U_q(-,+), \\ E_{21} & \leftrightsquigarrow & U_q(+,-), \\ E_{11} & \leftrightsquigarrow & U_q(-,-) = U_q(su(1,1)), \end{array}$$

where in $U_q(\mu,\nu)$,

$$[E,F] = \frac{\mu K^2 - \nu K^{-2}}{q-q^{-1}}.$$

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Associated partial coproducts:

$$\Delta^k_{i,j} : E_{ij} \to E_{ik} \otimes E_{kj} \leftrightsquigarrow \Delta^{\kappa}_{\mu,\nu} : U_q(\mu,\nu) \to U_q(\mu,\kappa) \otimes U_q(\kappa,\nu).$$

Fusion rules

Irreducible (unbounded) representations π_s^μ of $U_q(-,+)$ for $s \in \mathbb{R}$, $\mu \in \{-,+\}$.

Partial coproduct:

$$\Delta_{+,+}^{-} : U_q(su(2)) \to U_q(+,-) \otimes U_q(-,+).$$

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Irreducible (unbounded) representations π_s^μ of $U_q(-,+)$ for $s \in \mathbb{R}$, $\mu \in \{-,+\}$.

Partial coproduct:

$$\Delta^-_{+,+} : U_q(su(2)) \to U_q(+,-) \otimes U_q(-,+).$$

$\Rightarrow$ For irreducible *-representations π_s^μ and π_t^ν of $U_q(-,+)$: fusion

$$\pi_{s,t}^{\mu,\nu} : U_q(su(2)) \to B(\overline{\mathscr{H}_s^\mu} \otimes \mathscr{H}_t^\nu) : x \to ((\pi_s^\mu)^c \otimes \pi_t^\nu)\Delta^2_{11}(x).$$

Indeterminate moment problems

For Casimir $C \in U_q(su(2))$: $\pi_{s,t}^{\mu,\nu}(C)$ *not* essentially self-adjoint: eigenvectors in terms of q-inverse dual Hahn polynomials whose moment problem is indeterminate.

$\Rightarrow$ Decomposition of $\pi_{s,t}^{\mu,\nu}$ into irreducibles is not well-defined!

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$\Rightarrow$ Decomposition of $\pi_{s,t}^{\mu,\nu}$ into irreducibles is not well-defined!

However, on operator-algebraic level: we can paste certain representations together to get a good self-adjoint extension of certain direct sums of $\pi_{s,t}^{\mu,\nu}(C)$:

$$C \, \eta \, \mathscr{L}^{\infty}(\widehat{SU}_q(2))$$

and

$$\Delta_{11}^{2} : E_{22} \rightarrow E_{21} \otimes E_{12}.$$