Harmonic analysis on a locally compact hypergroup

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Outline

Definition and examples of a l.c. hypergroup.

Harmonic analysis on a l. c. hypergroup

Historical remarks.

Definition of a l.c. hypergroup.

Definition (Yu. Chapovsky, A. Kalyuzhny, G. Podkolzin' [1])

$(Q, \Delta, e, *, \mu)$ is a locally compact hypergroup if Q is a l. c. space, $*\colon Q \to Q$ is an involutive homeomorphism, $e^* = e \in Q$ and

(H_1) $\Delta\colon \mathcal{C}_b(Q) \to \mathcal{C}_b(Q \times Q)$ is a $\mathbb{C}$-linear mapping such that

(a) $(\Delta \times \mathrm{id}) \circ \Delta = (\mathrm{id} \times \Delta) \circ \Delta$;

(b) Δ is positive;

(c) $(\Delta 1)(p, q) = 1$, $\forall p, q \in Q$;

(d) $\forall f, g \in \mathcal{C}_c(Q)$, we have $(1 \otimes f) \cdot (\Delta g) \in \mathcal{C}_c(Q \times Q)$ and $(f \otimes 1) \cdot (\Delta g) \in \mathcal{C}_c(Q \times Q)$.

(H_2) If $\varepsilon(f) = f(e) : C_b(Q) \to \mathbb{C}$ then $(\varepsilon \times \mathrm{id}) \circ \Delta = (\mathrm{id} \times \varepsilon) \circ \Delta = \mathrm{id}$.

(H_3) The function $\check{f}$ defined by $\check{f}(q) = f(q^*)$ for $f \in \mathcal{C}_b(Q)$ satisfies $(\Delta \check{f})(p, q) = (\Delta f)(q^*, p^*)$.

[1] Yu. Chapovsky, A. Kalyuzhny, G. Podkolzin. Harmonic analysis on a locally compact hypergroup. *Methods of Func. Anal. and Topol.* **17** *(2011), 319–329.*

Definition (cont.)

(H_4) There exists a positive measure μ on Q, $\operatorname{supp}\mu = Q$, such that

$$\int_Q (\Delta f)(p,q)\, g(q)\, d\mu(q) = \int_Q f(q)\, (\Delta g)(p^*, q)\, d\mu(q)$$

for all $f \in \mathcal{C}_b(Q)$ and $g \in \mathcal{C}_c(Q)$, or $f \in \mathcal{C}_c(Q)$ and $g \in \mathcal{C}_b(Q)$, $p \in Q$; such a measure μ will be called a *left Haar measure* on Q.

Example

Let G be a l. c. group, $(\Delta f)(p,q) = f(pq)$, $p, q \in G$, $q^* = q^{-1}$, e be a neutral element. Then G is a l. c. hypergroup.

Example

Let G be a l. c. group, H be a compact subgroup of G with a Haar measure μ_H normalized by the condition $\int_H d\mu_H(p) = 1$. Let $Q = H \setminus G/H = \{HgH : g \in G\}$ be the set of double cosets endowed with the factor topology. Let

$$(\Delta f)(g_1, g_2) = \int_H f(g_1 h g_2) d\mu_H(h),\ f \in C_b(G),\ f(h_1 g h_2) = f(g)$$

be a comultiplication, $e = H$, $(HgH)^* = Hg^{-1}H$. Than Q is a l. c. hypergroup.

There are examples of l. c. hypergroups associated with Sturm–Liouville eqaution, with orthogonal polynomials, and so on.

Hypergroups constructed from a conditional expectation

Let A be a C^*-algebra and $B \subset A$ a C^*-subalgebra of A. A bounded linear map $P\colon A \to B$ is called a conditional expectation if it is a projection onto and has norm 1.

Theorem

Let Q be a l. c. hypergroup. Let $A = C_b(Q)$, $A_0 = C_0(Q)$ and I be an ideal of A consisting of functions with compact support. Let $P\colon A \to A$ be a conditional expectation such that $B = P(A_0)$ is a C^*-algebra, $P(I) \subset I$, $P(\check{f}) = \big(P(f)\big)\check{}$, and

$$\big((P \times \mathrm{id}) \circ \Delta \circ P\big)(f) = \big((\mathrm{id} \times P) \circ \Delta \circ P\big)(f) = \big((P \times P) \circ \Delta\big)(f),$$

for all $f \in A$.

Theorem (cont).

Denote by $\tilde{Q}$ the spectrum of the commutative algebra B. For each $g \in B$, let

$$\tilde{\Delta}(g) = \big((P \times P) \circ \Delta\big)(g).$$

If $\tilde{q} \in \tilde{Q}$ and $g \in B$, then we set $\tilde{q}^*(g) = \check{g}(q)$, $\tilde{e} = \varepsilon$, and let $\tilde{\mu}$ be defined by $\tilde{\mu} = \mu \circ P$. Then $(\tilde{Q}, *, \tilde{e}, \tilde{\Delta}, \tilde{\mu})$ is a locally compact hypergroup.

Hypergroups from a conditional expectation (cont.)

Definition

Let Q be a locally compact hypergroup. A conditional expectation P is called counital if $\varepsilon \circ P = \varepsilon$ on A.

Theorem

Let G be a l. c. group, $P\colon A \to A$ a conditional expectation satisfying the conditions of the previous theorem. Then there is a compact subgroup H of the group G and conditional expectations P_1, P_2 such that $P = P_2 \circ P_1$, where

$$(P_1 f)(g) = \int_H \int_H f(h_1 g h_2) d\mu_H(h_1) d\mu_H(h_2), \quad f \in C_b(G)$$

and P_2 is counital.

Banach algebra $L_1(Q, \mu)$

Convoluton of functions $f, g \in C_c(Q)$ and involution $f^\star$ are defined by the equalities

$$(f * g)(q) = \int_Q f(p)\,(\Delta g)(p^*, q)\,d\mu\,(p), \quad f^\star(q) = \bar{f}(q^*)\delta(q^*),$$

where $\delta(p)$ is a modular function of the hypergroup Q.

Theorem

The space $L_1(Q, \mu)$ is an involutive Banach algebra having an approximative identity.

A left Hilbert algebra structure

Definition

A linear subset $\mathfrak{A}$ of a Hilbert space H is called a left Hilbert algebra if $\mathfrak{A}$ is an associative algebra with involution $^\sharp$ and the following holds:

(i) the map $^\sharp\colon \mathfrak{A} \to \mathfrak{A}$ is a preclosed operator on H;

(ii) $\big(fg, h\big)_H = \big(g, f^\sharp h\big)_H$ for $f, g, h \in \mathfrak{A}$;

(iii) for every $f \in \mathfrak{A}$, the operator $L_f\colon g \mapsto fg$, $g \in \mathfrak{A}$, can be extended to a continuous operator on H;

(iv) $\mathfrak{A} \cdot \mathfrak{A}$ is dense in H.

Proposition

The algebra $\mathfrak{A} = \mathcal{C}_c(Q)$ with multiplication $*$, involution * and the scalar product induced from the Hilbert space $H = L_2(Q, \mu)$ is a left Hilbert algebra.

Representations of a l.c. hypergroup

Definition

Let H be a Hilbert space. A weakly continuous mapping $\pi \colon Q \to \mathcal{L}(H)$ is called a representation of Q if:

(i) $\pi(e) = I$;

(ii) $\pi(p^*) = \pi(p)^*$;

(iii) for every $\xi, \eta \in H$,

$$\Delta\big(\pi(\,\cdot\,)\xi, \eta\big)_H(p, q) = \big(\pi(p)\pi(q)\xi, \eta\big)_H.$$

Regular representations

For each $p \in Q$, let $L_p \colon \mathcal{C}_c(Q) \to \mathcal{C}_c(Q)$ and $R_p \colon \mathcal{C}_c(Q) \to \mathcal{C}_c(Q)$ be defined by

$$(L_p f)(q) = (\Delta f)(p^*, q), (R_p f)(q) = (\Delta f)(q, p)\, \delta^{\frac{1}{2}}(p).$$

Proposition

Let $H = L_2(Q, \mu)$. Then the mappings

$$\pi_L \colon p \mapsto L_p, \qquad \pi_R \colon p \mapsto R_p$$

are bounded representations of Q in H. Moreover, they separate points of Q.

Theorem

Bounded nondegenerate representations of the hypergroup Q are in one-to-one correspondence with nondegenerate representations of the Banach algebra $L_1(Q, \mu)$.

Definition

A continuous bounded function k on Q is called positive definite if $\forall n \in \mathbb{N}$, $\forall q_i \in Q$, $\forall \xi_i \in \mathbb{C}$, $i = 1, \ldots, n$ we have

$$\sum_{i,j=1}^{n} \xi_i \overline{\xi_j} \Delta k(q_i^*, q_j) \geq 0.$$

Definition

For two positive definite functions k_1, k_2, we say that k_1 majorizes k_2, written by $k_1 \succ k_2$, if $k_1 - k_2$ is positive definite. A positive definite function k is elementary if any positive definite function majorized by k is of the form λk, $\lambda \in [0, 1]$.

Theorem

A continuous function k on Q is bounded and positive definite if and only if there is a Hilbert space H_k, a bounded representation π_k of Q on H_k, and a vector $\xi_k \in H_k$ such that

$$k(q) = \left(\pi_k(q)\xi_k, \xi_k\right)_{H_k}, \qquad q \in Q.$$

The representation π_k is irreducible if and only if k is elementary.

Theorem

Every continuous function can be uniformly approximated on a compact set with linear combinations of elementary positive definite functions.

Theorem

Let $\mathcal{L}$ (resp. $\mathcal{R}$) denote the von Neumann algebra generated by the operators L_p (resp. R_p), $p \in Q$, on $H = L_2(Q, \mu)$. Then $\mathcal{L}' = \mathcal{R}$ and $\mathcal{R}' = \mathcal{L}$.

Plancherel theorem and invertion formula

Let $\mathfrak{A}$ be the left Hilbert algebra of a l. c. hypergroup Q. Let $H = L_2(Q, \mu)$ and denote by $(L_f(g))(p) = \int_Q f(q)(L_p g)(q)d\mu(q)$, $f \in \mathfrak{A}$, $g \in H$. Let $\mathcal{L}$ be the von Neumann algebra generated by L_f, $f \in \mathfrak{A}$ and φ the weight on $\mathcal{L}$ corresponding to the scalar product in H, i.e., it is defined by $\varphi(L_g^* L_f) = \big(f, g\big)_H$, $f, g \in \mathfrak{A}$. Let H_φ be the Hilbert space obtained from $\mathcal{L}$ and φ via the GNS-construction. The central decomposition theorem for von Neumann algebras applied to $\mathcal{L}$ gives

$$H_\varphi = \int_Z^\oplus H_\varphi(z)\, d\rho(z),\ \mathcal{L} = \int_Z^\oplus \mathcal{L}(z)\, d\rho(z),\ \varphi = \int_Z^\oplus \varphi_z\, d\rho(z),$$

where Z is the spectrum of the center of $\mathcal{L}$.

Plancherel theorem (cont.)

Definition

The measure ρ on Z will be called a Plancherel measure. The Fourier transform $\hat{f}$ of $f \in \mathfrak{A}$ is defined on Z by

$$\int_Q f(q)\, L_q(z)\, d\mu(q), \qquad z \in Z.$$

Theorem (L. Vainerman, G. Litvinov[2])

Let ρ is a Plancherel measure. Then

$$(f, g)_H = \int_Z \varphi_z\big(\hat{g}^*(z)\hat{f}(z)\big)\, d\rho(z),\ f(q) = \int_Z \varphi_z\big(L_q(z)^*\hat{f}(z)\big)\, d\rho(z),$$

and the Fourier transform $\hat{}$ can be extended to a unitary operator $L_2(Q, \mu) \to H_\varphi$.

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[2]L. Vainerman, G. Litvinov. Plancherel and invertion formulas for generalized translation operators. *Rep. Acad. Sci. USSR*, **257** *(1981), 792–795.*

Harmonic analysis on a cocommutative hypergroup

Definition

A hypergroup Q is called *cocommutative* if

$$\Delta f(p, q) = \Delta f(q, p),$$

for all $f \in \mathcal{C}_b(Q)$, $p, q \in Q$.

Definition

A function $\chi \in \mathcal{C}_b(Q)$ is called a *character* of the hypergroup Q if $(\Delta\chi)(p, q) = \chi(p)\chi(q)$ for all $p, q \in Q$. A character χ is called *Hermitian* if $\chi(p^*) = \overline{\chi(p)}$, $p \in Q$.

Let X_h be the space of bounded Hermitian characters endowed with the topology of the space of maximal ideals of the involutive Banach algebra $L_1(Q, \mu)$.

Harmonic analysis on a cocommutative hypergroup (cont.)

Theorem

A continuous function k on Q is positive definite if and only if it can uniquely be represented as an integral,

$$k(p) = \int_{X_h} \chi(p)\, d\nu(\chi),$$

with respect to some nonnegative finite Borel measure ν on the space X_h.

Definition

For a function $f \in L_1(Q, \mu)$, the function $\hat{f}: X_h \to \mathbb{C}$ defined by

$$\hat{f}(\chi) = \int_Q f(p)\bar{\chi}(p)\, d\mu(p),$$

is called the Fourier transform of f.

Harmonic analysis on a cocommutative hypergroup (cont.)

Proposition

The Fourier transform defines a unitary operator of the space $L_2(Q, \mu)$ onto the space $L_2(\hat{Q}, \rho)$, where ρ is the Plancherel measure on $\hat{X}_h$, $\hat{Q} = \operatorname{supp} \rho$, and the following inversion formula holds:

$$f(p) = \int_{\hat{Q}} \hat{f}(\chi)\chi(p)\, d\rho(\chi).$$

Theorem

Let Q be a cocommutative hypergroup satisfying the following properties:

(i) *the character ε defined in (H_2) belongs to $\hat{Q}$;*

(ii) *the product $\chi_1\chi_2$ of two characters $\chi_1, \chi_2 \in \hat{Q}$ is a positive definite function, and the support of the corresponding measure $\nu_{\chi_1\chi_2}$ is contained in $\hat{Q}$;*

Theorem (cont.)

(iii) the comultiplication $\hat{\Delta}\colon \mathcal{C}_b(\hat{Q}) \to \mathcal{C}_b(\hat{Q} \times \hat{Q})$ defined by

$$\hat{\Delta}(F)(\chi_1, \chi_2) = \int_{\hat{Q}} F(\chi)\, d\nu_{\chi_1\chi_2}(\chi),$$

$F \in \mathcal{C}_b(\hat{Q})$, satisfies axiom $(H_1)(\mathrm{d})$.

Then $\hat{Q}$ is also a locally compact commutative hypergroup, a so-called dual hypergroup, that satisfies the conditions of this theorem, and the hypergroup $\hat{\hat{Q}}$ dual to $\hat{Q}$ coincides with Q. The hypergroup dual to a compact hypergroup is a discrete hypergroup, the hypergroup dual to a discrete hypergroup is a compact hypergroup.

Historical remarks.

A family of generalized translation operators (Delsart' 1938[3], B. Levitan [4]) is a coalgebra structure on a space of functions.

Commutative hypercomplex systems with compact and discrete bases (Yu. Berezansky, S. Krein' 1950[5]) form a class of generalized convolution algebras with a rich harmonic analysis.

[3] J. Delsart. Sur une extension de la formule de Taylor. *J. Math. Pure at Appl.*, **17** *(1938), p. 213–231.*

[4] B. Levitan B. Theory of generalized translation operators. M.: Nauka, 1973 (in Russian)

[5] Yu. Berezansky Yu., Krein S. Continual algebras. *Rep. Acad. Sci USSR. bf 72 (1950), p.5–8 (in Russian).*

A DJS-hypergroup (C. Duncl' 1973 [6] R. Spector' 1975 [7] R. Jewett' 1975 [8]) is a generalized convolution measure algebra on a l. c. space Q admitting a rich harmonic analysis. It turned out that compact and discret commutative DJS-hypergroups make a subclass of hypercomplex systems with compact and discrete bases.

[6] C. Duncl. Structure hypergroups for measure algebras. *Pacific J. Math.*, **47** *(1973), 413 – 425.*

[7] R. Spector. Apercu de la theorie des hypergroups. *Lect. Notes in Math.*, **497** *(1975), 643 – 673.*

[8] R. Jewett. Spaces with an abstract convolution of measures. *Adv. in Math.*, **18** *(1975), 1 – 101.*

Unimodular hypercomplex systems with l. c. bases (Yu. Berezansky, A. Kalyuzhny' 1982[9]) are more general than the unimodular DJS-hypergroups, although the axiomatics of such hypercomplex systems is cumbersome and can not be generalized to a nonunimodular case.

Compact and discrete hypercomplex systems without the positivity condition (L. Vainerman' 1984 [10]) form a category with duality functor.

[9] Yu. Berezansky, A. Kalyuzhny. Harmonic analysis in hypercomples systems. Kluwer Acad. Publ., Dordecht–Boston–London, 1998, 483 p.

[10] L. Vainerman. Hypercomplex systems with compact and discrete bases. *Rep. of Acad. Sci. USSR*, **278** (1984), 16–20.