On the group of extensions for the bicrossed product construction for a locally compact group

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Historical remarks

A paper of G.I. Kac\(^1\) was the first one that considered the bicrossed construction and used it to construct Kac algebras. In this paper he gave a full description of the construction procedure for a matched pair of finite groups. It was shown by Kac that the bicrossed product construction can also be carried out with a pair of compatible 2-cocycles. Then this last construction was studied by S. Majid\(^2\) both in algebraic and analytic aspects. S. Baaj & G. Skandalis\(^3\) have defined a matched pair of Kac systems. In particular they considered a matched pair of locally compact groups.


Finaly the most general construction was given by S. Vaes & L. Vainerman\(^4\)

An important part in the bicrossed construction is an ability to construct compatible cocycles. It was found by Kac that the set of such cocycles forms a group. The structure of this group was studied by Kac, S. Baaj & G. Skandalis, S. Vaes & L. Vainerman\(^5\)


Locally compact quantum groups.

Definition (Kustermans, Vaes’ 2000\(^{6}\))

\(\mathcal{M} = (M, \Delta, \varphi, \psi)\) is a locally compact quantum group if

1. \(M\) is a von Neumann algebra;
2. \(\Delta : M \to M \otimes M\) is a normal unital \(*\)-homomorphism satisfying
   \[ (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta; \]
3. \(\varphi\) and \(\psi\) are normal semifinite faithful weights on \(M\), which are, respectively, left- and right-invariant, i.e.,
   \[ (\text{id} \otimes \varphi) \circ \Delta(a) = \varphi(a) 1, \]
   \[ (\psi \otimes \text{id}) \circ \Delta(b) = \psi(b) 1, \]
   where \(a \in M_{\varphi}^{+}\), \(b \in M_{\psi}^{+}\).

A matched pair of Lie groups

Definition
A pair of Lie groups, \((F, G)\), is called a \textit{matched pair} if there exists a Lie group \(K\) such that \(F < K\), \(G < K\), and \(F \cdot G = K\), \(F \cap G = \{e\}\).

Proposition
If \((F, G)\) is a matched pair of Lie groups, then there are a left action, \(\triangleright : G \times F \to F\), and a right action, \(\triangleleft : G \times F \to G\), defined by

\[ g \cdot f = (g \triangleright f) \cdot (g \triangleleft f), \quad (g \triangleright f) \in F, \ (g \triangleleft f) \in G. \]

Notations
For \(a \in C^\infty(F)\), \(b \in C^\infty(G)\), \(f \in F\), \(g \in G\), define

\[ (a \triangleleft g)(f) = a(g \triangleright f), \quad (f \triangleright b)(g) = b(g \triangleleft f). \]
Pairs of cocycles

**Definition**

A pair of $C^\infty$-functions $(u, v)$, where $u: G \times F \times F \to \mathbb{T}$ and $v: G \times G \times F \to \mathbb{T}$, is called a pair of cocycles for the matched pair $(F, G)$ if the function $h_{u,v}: K \times K \times K \to \mathbb{T}$ defined by

$$h_{u,v}(k_1, k_2, k_3) = u(g_1, f_2, g_2 \triangleright f_3) \cdot v(g_1 \lhd f_2, g_2, f_3),$$

$k_i = f_i g_i$, $i = 1, 2, 3$, is a reduced nonhomogeneous 3-cocycle on $K$.

Two pairs of cocycles $(u_1, v_1)$ and $(u_2, v_2)$ are called equivalent, if

$$h_{u_1,v_1} h_{u_2,v_2}^{-1} = d r$$

for some $C^\infty$-function $r: K \times K \to \mathbb{T}$ satisfying the condition

$$r(f_1 g_1, f_2 g_2) = r(g_1, f_2).$$
It is easy to check the following

**Fact**

Nonequivalent pairs of cocycles form a commutative group with respect to multiplication,

\[
[u_1, v_1] \cdot [u_2, v_2] = [u_1 \cdot u_2, v_1 \cdot v_2],
\]

which will be denoted by \( \text{Ext} (F, G) \).
The cocycle bicrossed product construction for Lie groups

Notations

Let \((F, G)\) be a matched pair of Lie groups and 
\([u, v] \in \text{Ext} (F, G)\).

For \(u: G \times F \times F \to \mathbb{T}\), define \(u_G: F \times F \to C^\infty(G, \mathbb{T})\) by

\[ u_G(f_1, f_2)(g) = u(g, f_1, f_2). \]

Notations

Define \(\mathcal{H} = L^2(F, \mu^l_F)\). For \(f \in F\), let \(l_f : \mathcal{H} \to \mathcal{H}\) be the left translation operator. Denote \(\mathcal{L}(F) = \{l_f \mid f \in F\}'' \subset B(\mathcal{H})\).
Proposition (G. I. Kac, S. Majid)

Define a von Neumann algebra,

\[ M_u = L^\infty(G) \otimes \mathcal{L}(F), \]

\[ (b_1 \otimes l_{f_1})(b_2 \otimes l_{f_2}) = b_1(f_1 \triangleright b_2)u_G(f_1, f_2) \otimes l_{f_1}l_{f_2}, \quad b_1, b_2 \in L^\infty(G), \quad l_{f_1}, l_{f_2} \in \mathcal{L}(F). \]

Identifying \( M_u \otimes M_u \simeq L^\infty(G \times G, \mathcal{L}(F) \otimes \mathcal{L}(F)) \) define \( \Delta_v : M_u \to M_u \otimes M_u \) by

\[ \Delta_v(b \otimes l_f)(g_1, g_2) = b(g_1g_2)\nu(g_1, g_2, f) l_{g_2 \triangleright f} \otimes l_f, \]

and \( \Phi(b \otimes l_f) = \delta_{f,e} \int_G b(g) d\mu_G^l(g), \)

\( \Psi(b \otimes l_f) = \delta_{f,e} \int_G b(g) \mu_G^r(g). \)

Then \( (M_u, \Delta_v, \Phi, \Psi) \) is a locally compact quantum group.
A description of $\text{Ext} (F, G)$

Proposition (Kac’ 1968, Baaj, Skandalis, Vaes’ 2005\(^7\))

There is a long exact sequence,

\[
\ldots \longrightarrow H^2(K, \mathbb{T}) \xrightarrow{\pi^2} H^2(F, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \xrightarrow{\sigma} \text{Ext} (F, G) \\
\xrightarrow{i} H^3(K, \mathbb{T}) \xrightarrow{\pi^3} H^3(F, \mathbb{T}) \oplus H^3(G, \mathbb{T}) \longrightarrow \ldots
\]

Theorem
Let \( \pi^i : H^i(K, \mathbb{T}) \longrightarrow H^i(F, \mathbb{T}) \oplus H^i(G, \mathbb{T}) \), 
\( i = 2, 3 \),
be defined by
\[
\pi^i = \pi^i_F + \pi^i_H,
\]
where \( \pi^i_F : H^i(K, \mathbb{T}) \rightarrow H^i(F, \mathbb{T}) \), \( \pi^i_G : H^i(K, \mathbb{T}) \rightarrow H^i(G, \mathbb{T}) \) are the restrictions. For
\[
[h_F + h_G] \in \left( H^2(F, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \right) / \pi^2(H^2(K, \mathbb{T})),
\]
define
\[
\kappa(h_F + h_G)(k_1, k_2, k_3) = h_F^{-1}(g_1 \triangleright f_2, (g_1 \lhd f_2)g_2 \triangleright f_3) \cdot h_F(f_2, g_2 \triangleright f_3) \cdot h_G(g_1 \lhd f_2(g_2 \triangleright f_3), g_2 \lhd f_3) \cdot h_G^{-1}(g_1 \lhd f_2, g_2),
\]
\( k_i = f_ig_i, \quad i = 1, 2, 3. \)
Theorem (continued)

For $h_K \in \text{Ker } \pi^3$, let

$$\xi(h_K) = h_K \cdot (d \cdot r)^{-1},$$

where $r \in C^\infty(K \times K)$ is defined by

$$r(k_1, k_2) = h_K(f_1, g_1, f_2g_2) \cdot h_K^{-1}(f_1, g_1 \triangleright f_2, (g_1 \triangleleft f_2)g_2) \cdot h_K^{-1}(g_1, f_2, g_2) \cdot h_K(g_1, f_2, g_2),$$

$$k_i = f_i g_i, \quad i = 1, 2.$$

Then

$$\bar{\kappa} \oplus \xi : \left( \left( H^2(F, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \right) / \text{Im } \pi^2 \right) \oplus \text{Ker } \pi^3 \longrightarrow \text{Ext } (F, G)$$

and is a group isomorphism.
**Example**

Let \( K = Z(3) \) be a group of upper triangular matrices (the Heisenberg group) \( K = \left\{ \begin{pmatrix} 1 & k_{12} & k_{13} \\ 0 & 1 & k_{23} \\ 0 & 0 & 1 \end{pmatrix} \right\} \),

\( F = \left\{ \begin{pmatrix} 1 & 0 & f_{13} \\ 0 & 1 & f_{23} \\ 0 & 0 & 1 \end{pmatrix} \right\}, \ G = \left\{ \begin{pmatrix} 1 & g_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \).

\( k_{ij}, : f_{ij}, g_{12} \in \mathbb{R} \).

Van Est isomorphism\(^8\) gives \( H^n(K, \mathbb{T}) \approx H^n(\mathfrak{k}, \mathbb{R}) \), \( H^n(G, \mathbb{T}) \approx H^n(\mathfrak{g}, \mathbb{R}) \), \( H^n(F, \mathbb{T}) \approx H^n(\mathfrak{f}, \mathbb{R}) \), where \( \mathfrak{k}, \mathfrak{g}, \) and \( \mathfrak{f} \) are the corresponding Lie algebras.

Example (continued)

It can be directly verified that

$$H^2(\mathfrak{k}, \mathbb{R}) = H^2(f, \mathbb{R}) \oplus H^2(g, \mathbb{R}).$$

And so $$\left( H^2(F, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \right) / \text{Im} \pi^2 \equiv 0.$$ Moreover, $$\dim(H^3(\mathfrak{k}, \mathbb{R})) = 1,$$ the corresponding left-invariant differential 3-form on $K$ is $\omega = dk_{12} \wedge dk_{23} \wedge dk_{13}$. Therefore a 3-coycle $h \in H^3(K, \mathbb{T})$ can be found as

$$h(k^1, k^2, k^3) = \exp(2i\pi \int_{\sigma(k^1,k^2,k^3)} \omega),$$

where $\sigma(k^1, k^2, k^3)(t_1, t_2, t_3) = \gamma_1(k^1 \gamma_{t_2} \gamma_{1-t_1-t_2} k^3)$ is the singular 3-simplex, $\gamma_t(k) = tk + (1 - t)e$ is curve in $K$ and $\Delta^3 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid 0 \leq t_1, t_2, t_3 \leq 1, \ t_1 + t_2 + t_3 \leq 1\}$ is the standard 3-simplex.
And corresponding pairs of cocycles are

\[ u(g^1, f^2, f^3) = \exp \left( 2i\pi \alpha g^1_{12} \det \begin{vmatrix} f^2_{23} & f^2_{13} \\ f^3_{13} & f^3_{13} \end{vmatrix} \right), \]

\[ v(g^1, g^2, f^3) = 1. \]
Matched pairs of Lie algebras

Definition
A pair \((\mathfrak{f}, \mathfrak{g})\) of Lie algebras is a matched pair (S. Majid), if there exists an algebra \(\mathfrak{k}\) such that \(\mathfrak{g}\) and \(\mathfrak{h}\) are Lie subalgebras of \(\mathfrak{k}\), \(\mathfrak{k} = \mathfrak{f} + \mathfrak{g}\) and \(\mathfrak{f} \cap \mathfrak{g} = \{0\}\).

Definition
A pair of linear functionals \((U, V)\), where \(U: \mathfrak{g} \otimes (\mathfrak{f} \wedge \mathfrak{f}) \to \mathbb{R}\)
\(V: (\mathfrak{g} \wedge \mathfrak{g}) \otimes \mathfrak{f} \to \mathbb{R}\) is called a pair of cocycles for the matched pair \((\mathfrak{f}, \mathfrak{g})\), if the linear functional \(F_{U,V}: \mathfrak{k} \wedge \mathfrak{k} \wedge \mathfrak{k} \to \mathbb{R}\), defined by

\[
F_{U,V}(A_1+X_1, A_2+X_2, A_3+X_3) = U(X_1; A_2, A_3) + U(X_2; A_3, A_1) + U(X_3; A_1, A_2) + V(X_1, X_2; A_3) + V(X_2, X_3; A_1) + V(X_3, X_1; A_2),
\]

\(A_i \in \mathfrak{f}, \ X_i \in \mathfrak{g}, \ i = 1, 2, 3,\) is an 3-cocycle on the algebra Lie \(\mathfrak{k}\).
Definition (continued)

Two pairs of cocycles $(U_1, V_1)$ and $(U_2, V_2)$ are called equivalent, if there exists a linear functional $R : \mathfrak{k} \wedge \mathfrak{k} \to \mathbb{R}$ such that

$$F_{U_1, V_1} - F_{U_2, V_2} = d R,$$

where $R(A_1, A_2) = R(X_1, X_2) = 0 \; \forall A_1, A_2 \in \mathfrak{f}, \; \forall X_1, X_2 \in \mathfrak{g}$. Here $d$ is the differentiation in the complex of multilinear antisymmetric forms on $\mathfrak{k}$.

As in the case of a matched pair of Lie groups, the classes of equivalent pairs $[U, V]$ of cocycles on the matched pair of Lie algebras form an Abelian group,

$[U_1, V_1] + [U_2, V_2] = [U_1 + U_2, V_1 + V_2]$. Denote this group by $\text{Ext}(\mathfrak{f}, \mathfrak{g})$. 
Lemma
Consider the maps $\lambda_{g_0} : K \to K$ and $\lambda_{f_0} : K \to K \ \forall f_0 \in F, \ \forall g_0 \in G$ defined by

$$\lambda_{f_0}(k) = (f_0 f) \cdot (g \triangleright f_0^{-1}), \quad \lambda_{g_0}(k) = (g_0 \triangleright f) \cdot (g g_0^{-1}); \quad k = fg.$$ 

And the map $\lambda_{k_0} : K \to K \ \lambda_{k_0} = \lambda_{f_0} \lambda_{g_0}, \ \forall k_0 = f_0g_0$. Then the map $\lambda : K \times K \to K$ is a left action of $K$ on itself.

Definition
A vector field $\eta : K \to T(K), \ \eta_k \in T_k(K)$, on the $K$ is called $\lambda$-invariant, if $\eta_{\lambda_{k_0}(k)} = (D\lambda_{k_0})_k \eta_k \ \forall k, k_0 \in K$, where $T(K)$ is a tangent bundle of $K$ and $(D\lambda_{k_0})_k : T_k(K) \to T_{\lambda_{k_0}(k)}(K)$ is the derivative of $\lambda_{k_0}$.

Notations
$\lambda$-invariant vector fields on $K$ form a Lie algebra $\tilde{\mathfrak{k}}$, isomorphic to $\mathfrak{k}$.
Explicit formula for cocycles for a matched pair of Lie groups

**Theorem**

Let \((F, G)\) be a matched pair of a connected, simply connected Lie groups, \((\mathfrak{f}, \mathfrak{g})\) a matched pair of the corresponding Lie algebras. Consider a \(\lambda\)-invariant differential form \(\omega_{U,V}\) on \(K\) corresponding to the pair of cocycles \([U, V] \in \text{Ext}(\mathfrak{f}, \mathfrak{g})\). There is a pair of singular 3-simplexes \(c^{2,1}(g_1, f_2, f_3)\) and \(c^{1,2}(g_1, g_2, f_3)\) on the group \(K\), the pair functions 
\[
\tilde{u} : F \times F \times G \to \mathbb{R} \quad \text{and} \quad \tilde{v} : G \times G \times F \to \mathbb{R},
\]
\[
\tilde{u}(g_1, f_2, f_3) = \int_{c^{2,1}(g_1,f_2,f_3)} \omega_{U,V}, \quad \tilde{v}(g_1, g_2, f_3) = \int_{c^{1,2}(g_1,g_2,f_3)} \omega_{U,V},
\]
such that the pair of functions
\[
u(g_1, f_2, f_3) = \exp (2i\pi \tilde{u}(g_1, f_2, f_3)),
\]
\[
u(g_1, g_2, f_3) = \exp (2i\pi \tilde{v}(g_1, g_2, f_3))\]
define a group homomorphism \(\text{Int} : \text{Ext}(\mathfrak{f}, \mathfrak{g}) \to \text{Ext}(F, G)\),
\[
\text{Int}([U, V]) = [u, v].
\]
Example
Let $F = \mathbb{R}^2$, $G = \mathbb{R}^2$ be a matched pair of Lie groups. Then $K = FG = \mathbb{R}^4$ is an Abelian group. We will denote by $f(a, b) \in F$ and $g(x, y) \in G$, $a, b, x, y \in \mathbb{R}$. $\mathfrak{k} = \mathfrak{f} + \mathfrak{g}$ is the corresponding Lie algebra. $A_1, A_2$ form a basis in $\mathfrak{f}$ and $X_1, X_2$ a basis in $\mathfrak{g}$. So non-equivalent pairs of cocycles are of the following form:

$$U(X_i; A_1, A_2) = \xi_i, \quad V(X_1, X_2; A_j) = \mu_j,$$

where $\xi_i, \mu_j \in \mathbb{R}$, $i, j = 1, 2$. The corresponding left-invariant forms on $K$ are given by

$$\omega_{U, V} = \xi_1 da \wedge db \wedge dx + \xi_2 da \wedge db \wedge dy +$$

$$+ \mu_1 da \wedge dx \wedge dy + \mu_2 db \wedge dx \wedge dy.$$
Example (continued)

**Proposition**

The functions $u(g; f_1, f_2) = \exp \left( 2i\pi (\xi_1 x + \xi_2 y) \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$, $v(g_1, g_2; f) = \exp \left( 2i\pi (\mu_1 a + \mu_2 b) \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \right)$ give pairs of cocycles for the matched pair of the Lie groups $(F, G)$.
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**Historical remarks**

**Definitions**

A description of the group $\text{Ext}(F, G)$

**Construction of a cocycle for double crossed products of Lie groups**

**Examples of cocycle bicrossed products of Lie groups**

$F = \{ \text{\textquotedbl}ax + b\text{\textquotedbl}, \ G = \mathbb{R}^2 \text{ with nontrivial actions.} \}$

**Example**

Let

$F = \{ f(a, b) = \begin{pmatrix} 1 & b & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \}, \ G = \{ g(x, y) = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \},$

$K = \{ k(a, b, x, y) = \begin{pmatrix} 1 & b & y \\ 0 & a & x \\ 0 & 0 & 1 \end{pmatrix} \},$ where

$a \in \mathbb{R} \setminus \{0\}, b, x, y \in \mathbb{R}$

Then the actions are defined as follows:

$g(x, y) \triangleright f(a, b) = f(a, b),$

$g(x, y) \lhd f(a, b) = g\left(\frac{x}{a}, y - \frac{x}{a}b\right) = \begin{pmatrix} 1 & 0 & y - \frac{x}{a}b \\ 0 & 1 & \frac{x}{a} \\ 0 & 0 & 1 \end{pmatrix}.$
Example (continued)

The corresponding left-invariant forms on $K$ are

$$\omega^1_v = \frac{1}{a} \left( - \frac{b}{a} da + db \right) \wedge dx \wedge dy,$$

$$\omega^2_v = \frac{1}{a^2} da \wedge dx \wedge dy.$$

Proposition

Two pairs of cocycles for the matched pair of the groups $(F, G)$ are defined as follow: $u^i = 1$, $i = 1, 2$, and

$$v^1(h_1, h_2; g) = \exp \left( 2i\pi \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \frac{b}{a} \right),$$

$$v^2(h_1, h_2; g) = \exp \left( 2i\pi \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \frac{a - 1}{a} \right),$$

where $h_i = h(x_i, y_i)$, $i = 1, 2$, and $g = g(a, b)$. 