On the group of extensions for the bicrossed product construction for a locally compact group

Podkolzin G. B.

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Historical remarks

A paper of G.I. KAC[1] was the first one that consided the bicrossed construction and used it to construct Kac algebras. In this paper he gave a full description of the construction procedure for a mathed pair of finite groups.
It was shown by Kac that the bicrossed product construction can also be carried out with a pair of compatible 2-cocycles. Then this last construction was studied by S.MAJID[2] both in algebraic and analytics aspects.
S. BAAJ & G. SKANDALIS[3] have defined a mathed pair of Kac systems. In particular they considered a mathed pair of locally compact groups.

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[1] G. I. Kac. Extensions of groups to ring groups. *Math. USSR Sbornik*, 5:451–474, 1968.

[2] S. Majid. *Foundations of quantum group theory.*Cambridge University Press, 1995.

[3] S. BAAJ & G. SKANDALIS, Transformations pentagonales. *C.R. Acad. Sci., Paris, Sér. I* **327** (1998), 623–628.

Finaly the most general construction was given by S. Vaes & L. Vainerman[4]

An important part in the bicrossed construction is an ability to construct compatible cocycles. It was found by Kac that the set of such cocycles forms a group.

The structure of this group was studied by Kac, S. Baaj & G. Skandalis, S. Vaes & L. Vainerman[5]

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[4] S. Vaes and L. Vainerman. Extensions of locally compact quantum groups and the bicrossed product construction. *Adv. in Math.*, 174(1):1–101, 2003.

[5] S. Vaes and L. Vainerman. On Low-Dimensional Locally Compact Quantum Groups *Locally Compact Quantum Groups and Groupoids. Proceedings of the Meeting of Theoretical Physicists and Mathematicians, Strasbourg, February 21 - 23, 2002., Ed. L. Vainerman, IRMA Lectures on Mathematics and Mathematical Physics, Walter de Gruyter, Berlin, New York (2003), pp. 127-187.*

Locally compact quantum groups.

Definition (Kustermans, Vaes' 2000[6])

$\mathfrak{M} = (M, \Delta, \varphi, \psi)$ is a locally compact quantum group if

1. M is a von Neumann algebra;
2. $\Delta : M \to M \otimes M$ is a normal unital $*$-homomorphism satisfying
$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta;$$
3. φ and ψ are normal semifinite faithful weights on M, which are, respectively, left- and right-invariant, i.e.,
$$(\mathrm{id} \otimes \varphi) \circ \Delta(a) = \varphi(a)\,1,$$
$$(\psi \otimes \mathrm{id}) \circ \Delta(b) = \psi(b)\,1,$$
where $a \in M_\varphi^+$, $b \in M_\psi^+$.

[6] J. Kustermans and S. Vaes. Locally compact quantum groups. *Ann. Scient. Ec. Norm. Sup.*, 33(6):837–934, 2000.

A matched pair of Lie groups

Definition

A pair of Lie groups, (F, G), is called a matched pair if there exists a Lie group K such that $F < K$, $G < K$, and
$F \cdot G = K, \qquad F \cap G = \{e\}$.

Proposition

If (F, G) is a matched pair of Lie groups, then there are a left action, $\triangleright : G \times F \to F$, and a right action, $\triangleleft : G \times F \to G$, defined by

$$g \cdot f = (g \triangleright f) \cdot (g \triangleleft f), \qquad (g \triangleright f) \in F, \ (g \triangleleft f) \in G.$$

Notations

For $a \in C^\infty(F)$, $b \in C^\infty(G)$, $f \in F$, $g \in G$, define

$$(a \blacktriangleleft g)(f) = a(g \triangleright f), \qquad (f \blacktriangleright b)(g) = b(g \triangleleft f).$$

Pairs of cocycles

Definition

A pair of C^∞-functions (u, v), where $u\colon G \times F \times F \to \mathbb{T}$ and $v\colon G \times G \times F \to \mathbb{T}$, is called a pair of cocycles for the matched pair (F, G) if the function $h_{u,v}\colon K \times K \times K \to \mathbb{T}$ defined by

$$h_{u,v}(k_1, k_2, k_3) = u(g_1, f_2, g_2 \triangleright f_3) \cdot v(g_1 \triangleleft f_2, g_2, f_3),$$

$k_i = f_i g_i$, $i = 1, 2, 3$, is a reduced nonhomogeneous 3-cocycle on K.

Two pairs of cocycles (u_1, v_1) and (u_2, v_2) are called equivalent, if

$$h_{u_1,v_1} h^{-1}_{u_2,v_2} = \mathrm{d}\, r$$

for some C^∞-function $r\colon K \times K \to \mathbb{T}$ satisfying the condition

$$r(f_1 g_1, f_2 g_2) = r(g_1, f_2).$$

It is easy to check the following

Fact

Nonequivalent pairs of cocycles form a commutative group with respect to multiplication,

$$[u_1, v_1] \cdot [u_2, v_2] = [u_1 \cdot u_2, v_1 \cdot v_2],$$

which will be denoted by $\mathrm{Ext}\,(F, G)$.

The cocycle bicrossed product construction for Lie groups

Notations

Let (F, G) *be a matched pair of Lie groups and* $[u, v] \in \mathrm{Ext}\,(F, G)$*.*
For $u\colon G \times F \times F \to \mathbb{T}$*, define* $u_G\colon F \times F \to C^\infty(G, \mathbb{T})$ *by*

$$u_G(f_1, f_2)(g) = u(g, f_1, f_2).$$

Notations

Define $\mathcal{H} = L^2(F, \mu_F^l)$*. For* $f \in F$*, let* $l_f : \mathcal{H} \to \mathcal{H}$ *be the left translation operator. Denote* $\mathcal{L}(F) = \{l_f \mid f \in F\}'' \subset B(\mathcal{H})$*.*

Proposition (G. I. Kac, S.Majid)

Define a von Neumann algebra,

$$M_u = L^\infty(G) \otimes \mathcal{L}(F),$$
$$(b_1 \otimes l_{f_1})(b_2 \otimes l_{f_2}) = b_1(f_1 \blacktriangleright b_2)u_G(f_1, f_2) \otimes l_{f_1 f_2},$$
$$b_1, b_2 \in L^\infty(G), \qquad l_{f_1}, l_{f_2} \in \mathcal{L}(F).$$

Identifying $M_u \otimes M_u \simeq L^\infty\big(G \times G, \mathcal{L}(F) \otimes \mathcal{L}(F)\big)$ *define* $\Delta_v \colon M_u \to M_u \otimes M_u$ *by*

$$\Delta_v(b \otimes l_f)(g_1, g_2) = b(g_1 g_2) v(g_1, g_2, f)\, l_{g_2 \triangleright f} \otimes l_f,$$

and
$$\Phi(b \otimes l_f) = \delta_{f,e} \int_G b(g)\, d\mu^l_G(g),$$
$$\Psi(b \otimes l_f) = \delta_{f,e} \int_G b(g)\, \mu^r_G(g).$$

Then $(M_u, \Delta_v, \Phi, \Psi)$ *is a locally compact quantum group.*

A description of $\mathrm{Ext}\,(F, G)$

Proposition (Kac' 1968, Baaj, Skandalis, Vaes' 2005[7])

There is a long exact sequence,

$$\ldots \longrightarrow H^2(K, \mathbb{T}) \xrightarrow{\pi^2} H^2(F, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \xrightarrow{\sigma} \mathrm{Ext}\,(F, G) \xrightarrow{i} H^3(K, \mathbb{T}) \xrightarrow{\pi^3} H^3(F, \mathbb{T}) \oplus H^3(G, \mathbb{T}) \longrightarrow \ldots$$

[7] S. Baaj, G. Skandalis, S. Vaes. Measurable Kac cohomology for bicrossed products, *Trans. Am. Math. Soc.*, **357**:4, pp. 1497–1524, 2005.

Theorem

Let $\pi^i\colon H^i(K,\mathbb{T}) \longrightarrow H^i(F,\mathbb{T}) \oplus H^i(G,\mathbb{T})$,
$i = 2, 3$,
be defined by

$$\pi^i = \pi^i_F + \pi^i_H,$$

where $\pi^i_F\colon H^i(K,\mathbb{T}) \to H^i(F,\mathbb{T})$, $\pi^i_G\colon H^i(K,\mathbb{T}) \to H^i(G,\mathbb{T})$ *are the restrictions. For*

$$[h_F + h_G] \in \Big(H^2(F,\mathbb{T}) \oplus H^2(G,\mathbb{T})\Big)/\pi^2\big(H^2(K,\mathbb{T})\big),$$

define

$$\begin{aligned}\bar{\kappa}(h_F + h_G)(k_1, k_2, k_3) \\ = h_F^{-1}(g_1 \triangleright f_2, (g_1 \triangleleft f_2)g_2 \triangleright f_3) \cdot h_F(f_2, g_2 \triangleright f_3)\cdot \\ h_G(g_1 \triangleleft f_2(g_2 \triangleright f_3), g_2 \triangleleft f_3) \cdot h_G^{-1}(g_1 \triangleleft f_2, g_2), \\ k_i = f_i g_i, \qquad i = 1, 2, 3.\end{aligned}$$

Theorem (continued)

For $h_K \in \mathrm{Ker}\,\pi^3$, let

$$\xi(h_K) = h_K \cdot (\mathrm{d}\, r)^{-1},$$

where $r \in C^\infty(K \times K)$ is defined by

$$r(k_1, k_2) = h_K(f_1, g_1, f_2 g_2) \cdot h_K^{-1}(f_1, g_1 \triangleright f_2, (g_1 \triangleleft f_2) g_2) \cdot \\ h_K^{-1}(g_1, f_2, g_2) \cdot h_K(g_1 \triangleright f_2, g_1 \triangleleft f_2, g_2), \\ k_i = f_i g_i, \qquad i = 1, 2.$$

Then

$$\bar{\kappa} \oplus \xi \colon \Big(\big(H^2(F, \mathbb{T}) \oplus H^2(G, \mathbb{T}) \big) / \mathrm{Im}\,\pi^2 \Big) \oplus \mathrm{Ker}\,\pi^3 \longrightarrow \\ \longrightarrow \mathrm{Ext}\,(F, G)$$

and is a group isomorphism.

Group $Z(3)$

Example

Let $K = Z(3)$ be a group of upper triangular matrices

(the Heisenberg group) $K = \{\begin{pmatrix} 1 & k_{12} & k_{13} \\ 0 & 1 & k_{23} \\ 0 & 0 & 1 \end{pmatrix}\},$

$F = \{\begin{pmatrix} 1 & 0 & f_{13} \\ 0 & 1 & f_{23} \\ 0 & 0 & 1 \end{pmatrix}\}, G = \{\begin{pmatrix} 1 & g_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}.$

$k_{ij}, : f_{ij}, g_{12} \in \mathbb{R}.$

Van Est isomorphism[8] gives $H^n(K, \mathbb{T}) \approx H^n(\mathfrak{k}, \mathbb{R})$, $H^n(G, \mathbb{T}) \approx H^n(\mathfrak{g}, \mathbb{R})$, $H^n(F, \mathbb{T}) \approx H^n(\mathfrak{f}, \mathbb{R})$, where $\mathfrak{k}$, $\mathfrak{g}$, and $\mathfrak{f}$ are the corresponding Lie algebras.

[8] A. Guichardet. Cohomologie des groupes topologiques et des algèbres de Lie *Cedic/Fernand Nathan, Paris, 1980.*

Example (continued)

It can be directly verified that

$$H^2(\mathfrak{k}, \mathbb{R}) = H^2(\mathfrak{f}, \mathbb{R}) \oplus H^2(\mathfrak{g}, \mathbb{R}).$$

And so $\Big(H^2(F, \mathbb{T}) \oplus H^2(G, \mathbb{T})\Big) / \operatorname{Im} \pi^2 \equiv 0.$ Moreover, $dim(H^3(\mathfrak{k}, \mathbb{R})) = 1$, the corresponding left-invariant differential 3-form on K is $\omega = dk_{12} \wedge dk_{23} \wedge dk_{13}$. Therefore a 3-coycle $h \in H^3(K, \mathbb{T})$ can be found as

$$h(k^1, k^2, k^3) = \exp(2i\pi \int_{\sigma(k^1,k^2,k^3)} \omega),$$

where $\sigma(k^1, k^2, k^3)(t_1, t_2, t_3) = \gamma_{t_1}\left(k^1 \gamma_{\frac{t_2}{1-t_1}}\left(k^2 \gamma_{\frac{t_3}{1-t_1-t_2}}(k^3)\right)\right)$ is the singular 3-simplex, $\gamma_t(k) = tk + (1-t)e$ is curve in K and $\Delta^3 = \{(t_1, t_2, t_3) \in \mathbb{R}^3 \mid 0 \leq t_1, t_2, t_3 \leq 1,\ t_1 + t_2 + t_3 \leq 1\}$ is the standard 3-simplex.

Example (continued)

And corresponding pairs of cocycles are

$$u(g^1, f^2, f^3) = \exp\left(2i\pi\alpha\, g^1_{12} det \begin{vmatrix} f^2_{23} & f^2_{13} \\ f^3_{13} & f^3_{13} \end{vmatrix}\right),$$

$$v(g^1, g^2, f^3) = 1.$$

Matched pairs of Lie algebras

Definition

A pair $(\mathfrak{f}, \mathfrak{g})$ of Lie algebras is a matched pair (S. Majid), if there exists an algebra $\mathfrak{k}$ such that $\mathfrak{g}$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{k}$, $\mathfrak{k} = \mathfrak{f} + \mathfrak{g}$ and $\mathfrak{f} \cap \mathfrak{g} = \{0\}$.

Definition

A pair of linear functionals (U, V), where $U : \mathfrak{g} \otimes (\mathfrak{f} \wedge \mathfrak{f}) \to \mathbb{R}$ $V : (\mathfrak{g} \wedge \mathfrak{g}) \otimes \mathfrak{f} \to \mathbb{R}$ is called a pair of cocycles for the matched pair $(\mathfrak{f}, \mathfrak{g})$, if the linear functional $F_{U,V} : \mathfrak{k} \wedge \mathfrak{k} \wedge \mathfrak{k} \to \mathbb{R}$, defined by

$$F_{U,V}(A_1+X_1, A_2+X_2, A_3+X_3) = U(X_1; A_2, A_3)+U(X_2; A_3, A_1)$$
$$+U(X_3; A_1, A_2)+V(X_1, X_2; A_3)+V(X_2, X_3; A_1)+V(X_3, X_1; A_2),$$

$A_i \in \mathfrak{f}$, $X_i \in \mathfrak{g}$, $i = 1, 2, 3$, is an 3-cocycle on the algebra Lie $\mathfrak{k}$.

Definition (continued)

Two pairs of cocycles (U_1, V_1) and (U_2, V_2) are called equivalent, if there exists a linear functional $R\colon \mathfrak{k} \wedge \mathfrak{k} \to \mathbb{R}$ such that

$$F_{U_1,V_1} - F_{U_2,V_2} = d\,R,$$

where $R(A_1, A_2) = R(X_1, X_2) = 0 \;\; \forall A_1, A_2 \in \mathfrak{f},\;\; \forall X_1, X_2 \in \mathfrak{g}$. Here d is the differentiation in the complex of multilinear antisymmetric forms on $\mathfrak{k}$.

As in the case of a matched pair of Lie groups, the classes of equivalent pairs $[U, V]$ of cocycles on the matched pair of Lie algebras form an Abelian group, $[U_1, V_1] + [U_2, V_2] = [U_1 + U_2, V_1 + V_2]$. Denote this group by $\mathrm{Ext}\,(\mathfrak{f}, \mathfrak{g})$.

Lemma

Consider the maps $\lambda_{g_0}\colon K \to K$ and $\lambda_{f_0}\colon K \to K$ $\forall f_0 \in F$, $\forall g_0 \in G$ defined by

$$\lambda_{f_0}(k) = (f_0 f) \cdot (g \triangleleft f_0^{-1}), \quad \lambda_{g_0}(k) = (g_0 \triangleright f) \cdot (g g_0^{-1}); \quad k = fg.$$

And the map $\lambda_{k_0}\colon K \to K$ $\lambda_{k_0} = \lambda_{f_0}\lambda_{g_0}$, $\forall k_0 = f_0 g_0$. Then the map $\lambda\colon K \times K \to K$ is a left action of K on itself.

Definition

A vector field $\eta\colon K \to T(K)$, $\eta_k \in T_k(K)$, on the K is called λ-invariant, if $\eta_{\lambda_{k_0}(k)} = (D\lambda_{k_0})_k \eta_k$ $\forall k, k_0 \in K$, where $T(K)$ is a tangent bundle of K and $(D\lambda_{k_0})_k\colon T_k(K) \to T_{\lambda_{k_0}(k)}(K)$ is the derivative of λ_{k_0}.

Notations

λ-invariant vector fields on K form a Lie algebra $\tilde{\mathfrak{k}}$, isomorphic to $\mathfrak{k}$.

Explicit formula for cocycles for a matched pair of Lie groups

Theorem

Let (F, G) be a mathed pair of a connected, simply connected Lie groups, $(\mathfrak{f}, \mathfrak{g})$ a mathed pair of the corresponding Lie algebras. Consider a λ–invariant differential form $\omega_{U,V}$ on K corresponding to the pair of cocycles $[U, V] \in \mathrm{Ext}\,(\mathfrak{f}, \mathfrak{g})$. There is a pair of singular 3-simplexes $c^{2,1}(g_1, f_2, f_3)$ and $c^{1,2}(g_1, g_2, f_3)$ on the group K, the pair functions $\tilde{u} : F \times F \times G \to \mathbb{R}$ and $\tilde{v} : G \times G \times F \to \mathbb{R}$,

$$\tilde{u}(g_1, f_2, f_3) = \int_{c^{2,1}(g_1, f_2, f_3)} \omega_{U,V}, \ \tilde{v}(g_1, g_2, f_3) = \int_{c^{1,2}(g_1, g_2, f_3)} \omega_{U,V},$$

such that the pair of functions
$u(g_1, f_2, f_3) = \exp\,(2i\pi\tilde{u}(g_1, f_2, f_3))$,
$v(g_1, g_2, f_3) = \exp\,(2i\pi\tilde{v}(g_1, g_2, f_3))$ *define a group homomorphism* $Int\colon \mathrm{Ext}\,(\mathfrak{f}, \mathfrak{g}) \to \mathrm{Ext}\,(F, G)$,
$Int\big([U, V]\big) = [u, v]$.

$F = \mathbb{R}^2$, $G = \mathbb{R}^2$ with trivial actions

Example

Let $F = \mathbb{R}^2$, $G = \mathbb{R}^2$ be a matched pair of Lie groups. Then $K = FG = \mathbb{R}^4$ is an Abelian group. We will denote by $f(a, b) \in F$ and $g(x, y) \in G$, $a, b, x, y \in \mathbb{R}$. $\mathfrak{k} = \mathfrak{f} + \mathfrak{g}$ is the corresponding Lie algebra. A_1, A_2 form a basis in $\mathfrak{f}$ and X_1, X_2 a basis in $\mathfrak{g}$. So non-equivalent pairs of cocycles are of the following form:

$$U(X_i; A_1, A_2) = \xi_i, \qquad V(X_1, X_2; A_j) = \mu_j,$$

where $\xi_i, \mu_j \in \mathbb{R}$, $i, j = 1, 2$. The corresponding left-invariant forms on K are given by

$$\omega_{U,V} = \xi_1 da \wedge db \wedge dx + \xi_2 da \wedge db \wedge dy + \\ + \mu_1 da \wedge dx \wedge dy + \mu_2 db \wedge dx \wedge dy.$$

Example (continued)

Proposition

The functions $u(g; f_1, f_2) = exp\left(2i\pi(\xi_1 x + \xi_2 y)\left|\begin{array}{cc} a_1 & a_2 \\ b_1 & b_2 \end{array}\right|\right)$, $v(g_1, g_2; f) = exp\left(2i\pi(\mu_1 a + \mu_2 b)\left|\begin{array}{cc} x_1 & x_2 \\ y_1 & y_2 \end{array}\right|\right)$ *give pairs of cocycles for the matched pair of the Lie groups* (F, G).

$F =$"$ax+b$", $G=\mathbb{R}^2$ with nontrivial actions.

Example

Let

$$F = \{f(a,b) = \begin{pmatrix} 1 & b & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}\},\ G = \{g(x,y) = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}\},$$

$$K = \{k(a,b,x,y) = \begin{pmatrix} 1 & b & y \\ 0 & a & x \\ 0 & 0 & 1 \end{pmatrix}\},\text{ where}$$

$a \in \mathbb{R} \setminus \{0\}, b, x, y \in \mathbb{R}$

Then the actions are defined as follows:

$g(x,y) \triangleright f(a,b) = f(a,b)$,

$$g(x,y) \triangleleft f(a,b) = g\left(\frac{x}{a}, y - \frac{x}{a}b\right) = \begin{pmatrix} 1 & 0 & y - \frac{x}{a}b \\ 0 & 1 & \frac{x}{a} \\ 0 & 0 & 1 \end{pmatrix}.$$

Example (continued)

The corresponding left-invariant forms on K are

$$\omega_v^1 = \frac{1}{a}\left(-\frac{b}{a}da + db\right) \wedge dx \wedge dy, \qquad \omega_v^2 = \frac{1}{a^2} da \wedge dx \wedge dy.$$

Proposition

Two pairs of cocycles for the matched pair of the groups (F, G) are defined as follow: $u^i = 1$, $i = 1, 2$, and

$$v^1(h_1, h_2; g) = exp\left(2i\pi \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \frac{b}{a}\right),$$

$$v^2(h_1, h_2; g) = exp\left(2i\pi \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \frac{a-1}{a}\right),$$

where $h_i = h(x_i, y_i)$, $i = 1, 2$, and $g = g(a, b)$.