II_1 factors with a unique Cartan decomposition

Operator algebras, Quantum groups and Tensor categories

In honor of *Leonid Vainerman*

Caen, 2012

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* Supported by ERC Starting Grant VNALG-200749

Collaboration with Leonid

- Extensions of locally compact quantum groups and the bicrossed product construction. *Advances in Mathematics* (2003).
- On low-dimensional locally compact quantum groups. Proceedings of the *2002 Strasbourg Conference*, organized by Leonid.
- Car driving in a one-way street in Bonn ... in the wrong direction.

Congratulations and Happy Birthday !

The group measure space construction

Input : a countable group Γ and a probability measure preserving action $\Gamma \curvearrowright (X, \mu)$. Consider $\Gamma \curvearrowright \mathrm{L}^\infty(X)$ by $(g \cdot F)(x) = F(g^{-1} \cdot x)$.

Output : the von Neumann algebra $M = \mathrm{L}^\infty(X) \rtimes \Gamma$.

- M is generated by a copy of $\mathrm{L}^\infty(X)$ and unitaries $(u_g)_{g \in \Gamma}$.
- We have $u_g u_h = u_{gh}$ and $u_g F u_g^* = g \cdot F$.
  Think of the semi-direct product of groups.
- M has a trace $\tau : M \to \mathbb{C}$ given by
  $\tau(F u_g) = 0$ if $g \neq e$ and $\tau(F) = \int F \, d\mu$.

Main research theme

Classify crossed products $\mathrm{L}^\infty(X) \rtimes \Gamma$ in terms of the group action data !

Freeness, ergodicity, II_1 factors

Fix a probability measure preserving (pmp) action $\Gamma \curvearrowright (X, \mu)$.

- The subalgebra $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ is maximal abelian if and only if $\Gamma \curvearrowright X$ is free : for all $g \neq e$ and for a.e. $x \in X$, we have $g \cdot x \neq x$.
- Assuming that $\Gamma \curvearrowright X$ is free, we get that $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a factor if and only if $\Gamma \curvearrowright X$ is ergodic : all Γ-invariant subsets of X have measure 0 or 1.

$\rightsquigarrow$ We only consider free ergodic pmp actions $\Gamma \curvearrowright (X, \mu)$.

Then, $\mathrm{L}^\infty(X) \rtimes \Gamma$ is a **II_1 factor.**

Examples of free ergodic pmp actions

- **Irrational rotation** $\mathbb{Z} \curvearrowright \mathbb{T}$ given by $n \cdot z = \exp(2\pi i \alpha n)\, z$ for a fixed irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.
- **Bernoulli action** $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ given by $(g \cdot x)_h = x_{hg}$.
- The action $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.
- The action $\Gamma \curvearrowright G/\Lambda$ for lattices $\Gamma, \Lambda < G$.
- Certain **profinite actions** $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_n$ with $[\Gamma : \Gamma_n] < \infty$.

$\rightsquigarrow$ They give all rise to II_1 factors $\mathrm{L}^\infty(X) \rtimes \Gamma$.

Cartan subalgebras

Definition

A Cartan subalgebra $A \subset M$ is a maximal abelian subalgebra whose normalizer $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M.

Examples :

- $\mathrm{L}^\infty(X) \subset \mathrm{L}^\infty(X) \rtimes \Gamma$ if $\Gamma \curvearrowright (X, \mu)$ is a free ergodic pmp action.
- $\mathrm{L}^\infty(X) \subset \mathrm{L}(\mathcal{R})$ if $\mathcal{R}$ is a countably infinite ergodic pmp equivalence relation on (X, μ).

  The previous example corresponds to the orbit equivalence relation.
- Generic case : $\mathrm{L}^\infty(X) \subset \mathrm{L}_\Omega(\mathcal{R})$ with a scalar 2-cocycle Ω.

Conclusion

Uniqueness of Cartan subalgebras = reducing the classification of $\mathrm{L}^\infty(X) \rtimes \Gamma$ to the classification of equivalence relations.

Amenable versus nonamenable

Connes' theorem (1975)

All amenable II_1 factors are isomorphic with the unique hyperfinite II_1 factor R.

- $L\Gamma \cong R$ for all amenable icc groups Γ.
- $L^\infty(X) \rtimes \Gamma \cong R$ for all free ergodic pmp actions of an infinite amenable group Γ.

Connes-Feldman-Weiss (1981)

All amenable II_1 equivalence relations are isomorphic with the unique hyperfinite II_1 equivalence relation.

$\rightsquigarrow$ The hyperfinite II_1 factor R has a unique Cartan subalgebra up to conjugacy by an automorphism: if $A, B \subset R$ are Cartan, there exists $\alpha \in \mathrm{Aut}(R)$ with $\alpha(A) = B$.

Non-uniqueness of Cartan subalgebras

Connes-Jones, 1981 : II_1 factors with at least two Cartan subalgebras that are non-conjugate by an automorphism.

$\rightsquigarrow$ Strongly ergodic actions $\Gamma \curvearrowright (X, \mu)$ such that $M = \mathrm{L}^\infty(X) \rtimes \Gamma$ is McDuff, i.e. $M \cong M \mathbin{\overline{\otimes}} R$.

$\rightsquigarrow$ Uniqueness of Cartan subalgebras seemed hopeless.

Voiculescu, 1995 : $\mathrm{L}\mathbb{F}_n$, $2 \leq n \leq \infty$, has no Cartan subalgebra.

Ozawa-Popa, 2007 : the II_1 factor $M = \mathrm{L}^\infty(\mathbb{Z}_p^2) \rtimes (\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}))$ has two non-conjugate Cartan subalgebras, namely $\mathrm{L}^\infty(\mathbb{Z}_p^2)$ and $\mathrm{L}(\mathbb{Z}^2)$.

Speelman-V, 2011 : II_1 factors with **many** Cartan subalgebras.

Main result

Theorem (Popa - V, 2011)

If $\mathbb{F}_n \curvearrowright (X, \mu)$ is an arbitrary free ergodic pmp action, then $\mathrm{L}^\infty(X)$ is the unique Cartan subalgebra of $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$, up to unitary conjugacy.

Consider wreath product groups $H \wr \Gamma = H^{(\Gamma)} \rtimes \Gamma$.

Corollary, using Gaboriau's work

- If H is an abelian group and $n \neq m$, then we have $\mathrm{L}(H \wr \mathbb{F}_n) \ncong \mathrm{L}(H \wr \mathbb{F}_m)$. Of course we need $H \neq \{e\}$.
- If $n \neq m$ and $\mathbb{F}_n \curvearrowright X$, $\mathbb{F}_m \curvearrowright Y$ are arbitrary free ergodic probability measure preserving actions, then $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n \ncong \mathrm{L}^\infty(Y) \rtimes \mathbb{F}_m$.

The first II_1 factors with unique Cartan subalgebra

From now on : unique Cartan = unique up to unitary conjugacy.

Ozawa-Popa, 2007

If $\mathbb{F}_n \curvearrowright (X, \mu)$ is a profinite free ergodic pmp action, then $\mathrm{L}^\infty(X)$ is the unique Cartan subalgebra of $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$.

Profinite ergodic action : $\Gamma \curvearrowright \varprojlim \Gamma/\Gamma_n$ where $[\Gamma : \Gamma_n] < \infty$.

First ingredient :
The complete metric approximation property of the free group, and of all its crossed products by profinite actions.

Second ingredient :
Popa's malleable deformation of any crossed product $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$.

$\rightsquigarrow$ We explain both ingredients, and their gradual improvements up to our main result, in the next slides.

Complete metric approximation property (CMAP)

Herz-Schur multipliers and CMAP (Haagerup, 1978)

- We call $f : \Gamma \to \mathbb{C}$ a Herz-Schur multiplier if the linear map $L\Gamma \to L\Gamma : u_g \mapsto f(g)u_g$ is completely bounded.

  We denote by $\|f\|_{cb}$ the cb-norm of that map.

- We say that Γ has CMAP if there exists a sequence of finitely supported Herz-Schur multipliers $f_n : \Gamma \to \mathbb{C}$ tending to 1 pointwise and satisfying $\limsup_n \|f_n\|_{cb} = 1$.

Examples :

- $\mathbb{Z}$ has CMAP by Fejér summation of Fourier series.
- Amenable groups have CMAP : the f_n can be taken positive definite.
- $\mathbb{F}_n$ has CMAP by suitably cutting off the maps $g \mapsto \rho^{|g|}$ where $0 < \rho < 1$, $\rho \to 1$.
- CMAP is stable under free products, direct products, and subgroups.

CMAP for II_1 factors and consequences

The notion of CMAP makes sense for II_1 factors :

- $L\Gamma$ has CMAP if and only if Γ has CMAP.
- If Γ has CMAP and $\Gamma \curvearrowright X$ is a **profinite** action, then $L^\infty(X) \rtimes \Gamma$ has CMAP.

Ozawa-Popa (2007) : if M has CMAP and $A \subset M$ is a Cartan subalgebra, then the inclusion $A \subset M$ is "weakly compact".

$\rightsquigarrow$ This roughly means that $\mathcal{N}_M(A) \curvearrowright A$ is almost a profinite action.

$\rightsquigarrow$ First step to study uniqueness of Cartan subalgebras for profinite actions of CMAP groups.

Popa's malleable deformations

Consider a crossed product $M = \mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$.

- For every $0 < \rho < 1$, we have a unital completely positive map $\psi_\rho : M \to M$ given by $\psi_\rho(bu_g) = \rho^{|g|} bu_g$ for all $b \in \mathrm{L}^\infty(X)$, $g \in \mathbb{F}_n$.
- One can dilate the family (ψ_ρ) into a **malleable deformation :** we have $M \subset \widetilde{M}$, together with a 1-parameter group of automorphisms $\alpha_t \in \mathrm{Aut}(\widetilde{M})$ such that $\psi_{\rho_t}(x) = E_M(\alpha_t(x))$ for all $x \in M$.

Ozawa-Popa (2007) : if $A \subset M$ is a weakly compact Cartan subalgebra, the malleable deformation can be used to prove that A must be unitarily conjugate with $\mathrm{L}^\infty(X)$.

$\rightsquigarrow$ Uniqueness of Cartan subalgebras for profinite crossed products $\mathrm{L}^\infty(X) \rtimes \mathbb{F}_n$ follows.

More groups with deformations

Free group $\mathbb{F}_n$: the function $g \mapsto |g|$ is conditionally of negative type and proper. $\rightsquigarrow$ $g \mapsto \rho^{|g|}$ for a fixed $0 < \rho < 1$ is positive definite, and tends to 1 pointwise if $\rho \to 1$.

Most general conditionally negative type function on a countable group Γ : functions of the form $g \mapsto \|c(g)\|^2$,

where $c : \Gamma \to H_{\mathbb{R}}$ is a **1-cocycle** into the orthogonal representation $\pi : \Gamma \to \mathcal{O}(H_{\mathbb{R}})$, i.e. a map satisfying $c(gh) = c(g) + \pi(g)c(h)$.

Theorem (Ozawa-Popa, 2008)

Assume that Γ is nonamenable, has CMAP and that

- Γ admits a proper 1-cocycle into an orthogonal representation that is weakly contained in the regular representation,
- $\Gamma \curvearrowright (X, \mu)$ is a profinite action,

then $\mathrm{L}^\infty(X)$ is the unique Cartan subalgebra of $\mathrm{L}^\infty(X) \rtimes \Gamma$.

Weak amenability

Definition (Cowling-Haagerup, 1988)

A group Γ is weakly amenable if there exists a sequence of finitely supported Herz-Schur multipliers $f_n \to \mathbb{C}$ tending to 1 pointwise such that $\limsup_n \|f_n\|_{\mathrm{cb}} < \infty$.

(The optimal value of this lim sup is the Cowling-Haagerup constant of Γ.)

Ozawa, 2010 : in all the previous results, CMAP may be replaced by weak amenability.

If $\Gamma \curvearrowright X$ is a profinite action and $M = \mathrm{L}^\infty(X) \rtimes \Gamma$, we still have that any Cartan subalgebra $A \subset M$ is "weakly compact".

Importance of this improvement : all Gromov hyperbolic groups are weakly amenable (Ozawa, 2007).

Coarse 1-cocycles

Recall : a 1-cocycle of Γ into an orthogonal representation $\pi : \Gamma \to \mathcal{O}(H_{\mathbb{R}})$ is a map $c : \Gamma \to H_{\mathbb{R}}$ satisfying $c(gh) = c(g) + \pi(g)c(h)$ for all $g, h \in \Gamma$.

Coarse 1-cocycles : consider a map $c : \Gamma \to H_{\mathbb{R}}$ that only satisfies $\sup_{k\in\Gamma} \|c(gkh) - \pi(g)c(k)\| < \infty$ for all $g, h \in \Gamma$.

Theorem (Chifan-Sinclair, 2011), applicable to all hyperbolic groups

Assume that Γ is a nonamenable, weakly amenable group and that

- Γ admits a proper coarse 1-cocycle into an orthogonal representation that is weakly contained in the regular representation,
- $\Gamma \curvearrowright (X, \mu)$ is a profinite action,

then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$.

Coarse 1-cocycles, class $\mathcal{S}$, hyperbolic groups

Recall : a coarse 1-cocycle of Γ into an orthogonal rep $\pi : \Gamma \to \mathcal{O}(H_{\mathbb{R}})$ is a map $c : \Gamma \to H_{\mathbb{R}}$ s.t. $\sup_{k\in\Gamma} \|c(gkh) - \pi(g)c(k)\| < \infty$ for all $g, h \in \Gamma$.

Ozawa's class $\mathcal{S}$: we say that Γ is in the class $\mathcal{S}$ if Γ admits a compactification $\Gamma \subset K$ such that the left-right action $\Gamma \times \Gamma \curvearrowright \Gamma$ extends to an action by homeomorphisms of K with

- the left action $\Gamma \curvearrowright K$ being topologically amenable,
- the right action $\Gamma \curvearrowright K$ being trivial on $K - \Gamma$.

Example : hyperbolic groups are in $\mathcal{S}$ by taking the Gromov boundary.

Theorem (Ozawa, Chifan-Sinclair, 2011)

$\Gamma \in \mathcal{S}$ if and only if Γ is exact and admits a proper coarse 1-cocycle into a representation that is weakly contained in the regular representation.

Unique Cartan for arbitrary actions

Novelty of Popa-V approach : we discovered the correct notion of "weak compactness relative to $\mathrm{L}^\infty(X)$" for a Cartan subalgebra $A \subset \mathrm{L}^\infty(X) \rtimes \Gamma$.

Theorem (Popa-V, 2011-2012)

Let Γ be a nonamenable, **weakly amenable** group in **class $\mathcal{S}$** and let $\Gamma \curvearrowright (X, \mu)$ be an **arbitrary** free ergodic pmp action.

Then $\mathrm{L}^\infty(X)$ is the unique Cartan subalgebra of $\mathrm{L}^\infty(X) \rtimes \Gamma$.

The theorem applies to

- all non-elementary hyperbolic groups,
- all nonamenable discrete subgroups of a rank one simple Lie group,
- all limit groups in the sense of Sela.

The theorem also holds for direct products of such groups.

Unique Cartan in the type III case

The same result holds for **nonsingular actions** $\Gamma \curvearrowright (X, \mu)$.

$\rightsquigarrow$ Crossed product $L^\infty(X) \rtimes \Gamma$ can be of any type I, II or III.

Theorem (Houdayer-V, 2012)

Let Γ be a **weakly amenable** group in **class $\mathcal{S}$** and let $\Gamma \curvearrowright (X, \mu)$ be an **arbitrary** free ergodic **nonsingular** action.

Then either $L^\infty(X) \rtimes \Gamma$ is amenable, or $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$.

Note: $L^\infty(X) \rtimes \Gamma$ can be amenable without Γ being amenable.

$\mathcal{C}$-rigid groups

$\mathcal{C}$-rigid groups : groups Γ such that all $L^\infty(X) \rtimes \Gamma$ have unique Cartan.

Conjecture

If $\beta_n^{(2)}(\Gamma) \neq 0$ for some $n \geq 1$, then Γ is $\mathcal{C}$-rigid.

Supporting evidence :

- (Popa-V, 2011) All weakly amenable Γ with $\beta_1^{(2)}(\Gamma) > 0$ are $\mathcal{C}$-rigid.

$\mathcal{C}_{\text{gms}}$-rigid groups : all $L^\infty(X) \rtimes \Gamma$ have a unique **group measure space** Cartan subalgebra.

- (Popa-V, 2009) All $\Gamma_1 * \Gamma_2$ with Γ_1 infinite property (T) are $\mathcal{C}_{\text{gms}}$-rigid.
- (Chifan-Peterson, 2010) If $\beta_1^{(2)}(\Gamma) > 0$ and if Γ has a nonamenable subgroup with the relative property (T), then Γ is $\mathcal{C}_{\text{gms}}$-rigid.
- (Ioana, 2011) If $\beta_1^{(2)}(\Gamma) > 0$ and if $\Gamma \curvearrowright (X, \mu)$ is either rigid or compact, then $L^\infty(X) \rtimes \Gamma$ has a unique group measure space Cartan.

Which groups are $\mathcal{C}$-rigid ?

$\rightsquigarrow$ A characterization seems even difficult go guess !

- Conjecturally : if $\beta_n^{(2)}(\Gamma) > 0$, then Γ is $\mathcal{C}$-rigid.
- (Ozawa-Popa '08, Popa-V '09)
  If $\Gamma = H \rtimes \Lambda$ with H infinite abelian, then Γ is **not $\mathcal{C}$-rigid.**
- **Question :** give an example of a group Γ that has no almost normal infinite amenable subgroup and that is not $\mathcal{C}$-rigid either !

Still hungry ?

- More mathematical details : séminaire Algèbres d'Opérateurs à Paris (Chevaleret), jeudi 29 mars, 14h00.
- Otherwise : have a nice lunch !