Weak Multiplier Hopf Algebras.

Integrals and duality.

A. Van Daele

Department of Mathematics
University of Leuven

March 2012 / Caen

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- Weak multiplier Hopf algebras.

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This is about joint work in progress with Shuanhong Wang from Southeast University Nanjing (China).

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For a weak multiplier Hopf algebra, the canonical maps are no longer assumed to be bijective.

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There is a unique antipode S giving 'generalized inverses' of the canonical maps. It is a linear map $S : A \to M(A)$ and it is both a anti-algebra and a anti-coalgebra map.

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- The kernels of the canonical maps are given by the ranges of the idempotents $1-F_1$ and $1-F_2$ respectively where F_1 and F_2 are obtained as follows.

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The idempotent elements F_1 and F_2

Let (A, Δ) and E in $M(A \otimes A)$ be as before.

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There exists a right multiplier F_1 of $A \otimes A^{op}$ and a left multiplier F_2 of $A^{op} \otimes A$, uniquely determined by

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Remark

- *These idempotents F_1 and F_2 define idempotent maps G_1 and G_2 from $A \otimes A$ to itself by*

$$G_1(a \otimes b) = (a \otimes 1)F_1(1 \otimes b)$$
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- *We have $T_1 \circ (1 - G_1) = 0$ and $T_2 \circ (1 - G_2) = 0$.*

Existence of the antipode

Definition

A generalized inverse R_1 of T_1 is a linear map from $A \otimes A$ to itself so that $T_1 R_1 T_1 = T_1$ and $R_1 T_1 R_1 = R_1$. Similarly for T_2.

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Proposition

There exists a unique linear map S from A to $M(A)$, such that The maps R_1 and R_2 given by

$$R_1(a \otimes b) = \sum_{(a)} a_{(1)} \otimes S(a_{(2)})b$$
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are generalized inverses of the canonical maps T_1 and T_2.

Properties of the antipode

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- *If the map* S *is bijective from* A *to itself, we call the weak multiplier Hopf algebra regular. This happens, as in the case of Hopf algebras, precisely if flipping the coproduct on* A *(or the multiplication in* A*) still yields a weak multiplier Hopf algebra.*

The source and target maps

Recall that in a groupoid, the product pq of two elements p, q is defined if the source $s(p)$ is equal to the target $t(p)$.

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Assume that (A, Δ) is a weak multiplier Hopf algebra with antipode S. The source and target maps ε_s and ε_t are defined as

$$\varepsilon_t(a) = \sum a_{(1)}S(a_{(2)}) \qquad \text{and} \qquad \varepsilon_s(a) = \sum S(a_{(1)})a_{(2)}.$$

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- *The algebras* A_s *and* A_t *(or rather* $\varepsilon_s(A)$ *and* $\varepsilon_t(A)$*) are the 'left' and the 'right' leg of* E *and* $E \in M(\varepsilon_s(A) \otimes \varepsilon_t(A))$.

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- If (A, Δ) is a multiplier Hopf algebra, the algebras A_s and A_t are scalar multiples of the identity, and we get the usual definitions.
- As the antipode S flips the coproduct and maps A_t to A_s, it will map left integrals to right integrals (and vice versa).

Properties of integrals on WMHA's

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The proof is rather tricky.

More properties of integrals

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- *This result looks completely the same as for multiplier Hopf algebras (and algebraic quantum hypergroups).*
- *It is the main reason why many properties for multiplier Hopf algebras with integrals remain true for weak multiplier Hopf algebras.*

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If one of the components has an integral and the other has not, we get a weak multiplier Hopf algebra with an integral, but not with a faithful one.

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- We are working on an algebroid approach. This should not be a problem.
- We need to give more and interesting examples and possible applications.
- We need to study the case of a weak multiplier Hopf *-algebra with positive integrals and relate this with the work on measured quantum groupoids.

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(Preprints K.U. Leuven and Southeast University of Nanjing).