Multiplicative unitary for quantum codouble

S.L. Woronowicz

Caen
21.03.2012

Colour test

These slides are not for Daltonists
(colorblind persons).

czerwony
zielony
niebieski
zolty
fioletowy
magneta
purpurowy
brazowy

Leg numbering notation

$$(a \otimes b)_{12} = a \otimes b \otimes I,$$
$$(a \otimes b)_{23} = I \otimes a \otimes b,$$
$$(a \otimes b)_{13} = a \otimes I \otimes b.$$

This notation extends (by linearity and strong continuity) to all operators acting on $\mathcal{H} \otimes \mathcal{H}$

Definition

Let W be a unitary operator acting on $\mathcal{H} \otimes \mathcal{H}$. We say that W is a multiplicative unitary if the following pentagon equation

$$W_{23}W_{12} = W_{12}W_{13}W_{23}$$

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Example

G - a locally compact topological group,
$\mathcal{H}$ - a space of functions on G,
$\mathcal{H} \otimes \mathcal{H}$ - a space of functions on $G \times G$,

$$(Wx)(g,h) = x(gh,h).$$

Then

$$(W_{23}W_{12}x)(g,h,k) = x(g(hk),hk,k)$$
$$(W_{12}W_{13}W_{23}x)(g,h,k) = x((gh)k,hk,k)$$

$$\begin{pmatrix} \text{PENTAGON} \\ \text{EQUATION} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} g(hk) = (gh)k \\ \text{for all } g,h,k \in G \end{pmatrix}.$$

W is unitary iff $\mathcal{H}$ is the space of square integrable functions with respect to the right Haar measure.

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Quantum groups and multiplicative unitaries

$$G = (A, \Delta)$$

A is a non-degenerate subalgebra of $B(\mathcal{H})$,
$\Delta \in \mathrm{Mor}(A, A \otimes A)$, Δ - coassociative.
W is a unitary operator acting on $\mathcal{H} \otimes \mathcal{H}$.

W is a multiplicative unitary for G if

- $A = \{(\omega \otimes \mathrm{id})W : \omega \in B(\mathcal{H})_*\}^{\mathrm{CLS}}$
- $\Delta(a) = W(a \otimes I)W^*$ for any $a \in A$
- $(\mathrm{id} \otimes \Delta)W = W_{12}W_{13}$

Then

$$W_{23}W_{12} = W_{12}W_{13}W_{23}$$

Transposition of operators

Let $\overline{\mathcal{H}}$ be the Hilbert space complex-congugate to $\mathcal{H}$. Then we have an antilinear isometric bijection

$$\mathcal{H} \ni x \longleftrightarrow \overline{x} \in \overline{\mathcal{H}}$$

and linear antimultiplicative preserving hermitian conjugation bijection (called transposition)

$$B(\mathcal{H}) \ni a \longleftrightarrow a^{\top} \in B(\overline{\mathcal{H}})$$

such that $a^{\top}\overline{x} = \overline{a^{*}x}$ and $\left(\overline{x}\middle|a^{\top}\middle|\overline{y}\right) = (y|a|x)$ for all $x, y \in \mathcal{H}$ and $a \in B(\mathcal{H})$. Transposition is also defined for closed (densely defined) operators. In particular $\mathcal{D}(a^{\top}) = \overline{\mathcal{D}(a^{*})}$.

Manageability

Definition

Multiplicative unitary $W \in B(\mathcal{H} \otimes \mathcal{H})$ *is called manageable if there exist unitary* $\widetilde{W} \in B(\overline{\mathcal{H}} \otimes \mathcal{H})$ *and strictly positive selfadjoint* Q *acting on* $\mathcal{H}$ *such that*

- $W(Q \otimes Q)W^* = Q \otimes Q$
- $\left(x \otimes y \middle| W \middle| z \otimes u\right) = \left(\overline{z} \otimes Qy \middle| \widetilde{W} \middle| \overline{x} \otimes Q^{-1}u\right)$

  for all $x, z \in \mathcal{H}$, $y \in \mathcal{D}(Q)$ *and* $u \in \mathcal{D}(Q^{-1})$.

Theorem

Let $\mathcal{K}(\mathcal{H})$ *denote the algebra of all compact operators acting on* $\mathcal{H}$. *If* W *is manageable then*

$$W \in M(\mathcal{K}(\mathcal{H}) \otimes A).$$

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Scaling group

Theorem

If W is manageable then there exists a one-parameter group $(\tau_t)_{t\in\mathbb{R}}$ of A such that

$$\tau_t(a) = Q^{2it} a Q^{-2it}$$

for any $a \in A$ and $t \in \mathbb{R}$. Moreover

$$\Delta\circ\tau_t = (\tau_t \otimes \tau_t)\circ\Delta$$

Analytic generator $\tau_{i/2}$

Let $a, b \in A$. Then

$$\begin{pmatrix} a \in \mathcal{D}(\tau_{i/2}) \\ b = \tau_{i/2}(a) \end{pmatrix} \Longleftrightarrow \left(aQ \subset Qb \right)$$

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Unitary antipode

Theorem

If W is manageable then

- *There exists an antiautomorphism*

$$A \ni a \longmapsto a^R \in A$$

  such that

$$\widetilde{W} = W^{\top \otimes R}.$$

- $\Delta(a^R) = \Delta^{\mathrm{op}}(a)^{R \otimes R}$
- *R commutes with all τ_t*

Antipode

Theorem

Let W be manageable and $\kappa = R \circ \tau_{i/2}$. Then

- κ *is an unbounded linear operator acting on* A.
- $\{(\omega \otimes \mathrm{id})W : \omega \in B(\mathcal{H})_*\}$ *is a core for* κ *and* $\kappa\left((\omega \otimes \mathrm{id})W\right) = (\omega \otimes \mathrm{id})\left(W^*\right)$.
- $\mathcal{D}(\kappa)$ *is a subalgebra of* A *and* $\kappa(ab) = \kappa(b)\kappa(a)$ *for any* $a, b \in \mathcal{D}(\kappa)$.
- $\kappa(a)^* \in \mathcal{D}(\kappa)$ *and* $\kappa\left(\kappa(a)^*\right)^* = a$ *for any* $a \in \mathcal{D}(\kappa)$

Duality

$$\mathrm{flip}(a \otimes b) = (a \otimes b)_{21} = b \otimes a.$$

Let $W \in B(\mathcal{H} \otimes \mathcal{H})$ be a manageable multiplicative unitary and

$$\widehat{W} = W^*_{21}.$$

Then $\widehat{W}$ is a multiplicative unitary. Manageable with

$$\widehat{Q} = Q \text{ and } \widetilde{\widehat{W}} = \widehat{W}_{21}^{*(\top\otimes\top)}.$$

In what follows we denote by $\widehat{A}$, $\widehat{\Delta}$, $\widehat{\tau}$, $\widehat{R}$, ... the C^*-algebra, comultiplication, scaling group, unitary antipode, ... related to $\widehat{W}$:

$$\widehat{A} = \{(\mathrm{id} \otimes \omega)W^* : \omega \in B(\mathcal{H})_*\},$$

$$\widehat{\Delta}(\widehat{a}) = (W^*(I \otimes \widehat{a})W)_{21}.$$

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Duality

Theorem

$$W \in M(\widehat{A} \otimes A)$$
$$(\widehat{\Delta} \otimes \mathrm{id})W = W_{23}W_{13},$$
$$(\widehat{\tau}_t \otimes \tau_t)W = W,$$
$$W^{\widehat{R} \otimes R} = W.$$

Double group construction

Twisted flip

$$\sigma \in \mathrm{Mor}\left(A \otimes \widehat{A}, \widehat{A} \otimes A\right):$$

$$\sigma(a \otimes \widehat{a}) = W(\widehat{a} \otimes a)W^*.$$

Construction

$$\mathcal{A} = A \otimes \widehat{A},$$

$$\Delta = (\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \circ (\Delta \otimes \widehat{\Delta}).$$

Then $\Delta \in \mathrm{Mor}(\mathcal{A}, \mathcal{A} \otimes \mathcal{A})$, Δ is coassociative.

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Problem

Find multiplicative unitary for (A, Δ).

T. Yamanouchi (2000) and T. Masuda, Y. Nakagami and SLW (2003) found the formula assuming the existence of the Haar weights. Their formulae use in an essential way the operators J and $\widehat{J}$ (of Tomita-Takesaki theory) related to the Haar weights for the original group and its dual.

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Old formulae

Yamanouchi (2000)

$$\mathbb{W} = ((\hat{J} \otimes J) W (\hat{J} \otimes J))_{12} \widehat{W}_{23} \widehat{W}_{13} ((\hat{J} \otimes J) W^* (\hat{J} \otimes J))_{12} W_{24}.$$

Masuda Nakagami SLW (2003)

$$\mathbb{W} = \widehat{W}_{23} \widehat{W}_{13} ((J \otimes J) \widehat{W}^* (J \otimes J))_{23} W_{24}.$$

The main formula

Consider operator $\mathbb{W}$ acting on $\overline{H} \otimes H \otimes H \otimes \overline{H} \otimes H \otimes H$ introduced by

$$\mathbb{W}_{012345} = W_{24} W_{14} \widehat{W}_{25} W_{04}^{\widehat{R}\top\otimes\mathrm{id}} \widehat{W}_{05}^{R\top\otimes\mathrm{id}}. \tag{1}$$

Then $\mathbb{W} \in B(K \otimes K)$ (where $K = \overline{H} \otimes H \otimes H$) is a multiplicative unitary.

Pentagon equations

$$W_{\beta\gamma} W_{\alpha\beta} W^*_{\beta\gamma} = W_{\alpha\beta} W_{\alpha\gamma}, \tag{2}$$

$$W_{\beta\gamma} W^{\widehat{R}\top\otimes\mathrm{id}}_{\alpha\beta} W^*_{\beta\gamma} = W^{\widehat{R}\top\otimes\mathrm{id}}_{\alpha\beta} W^{\widehat{R}\top\otimes\mathrm{id}}_{\alpha\gamma}, \tag{3}$$

$$\widehat{W}_{\beta\gamma} \widehat{W}_{\alpha\beta} \widehat{W}^*_{\beta\gamma} = \widehat{W}_{\alpha\beta} \widehat{W}_{\alpha\gamma}, \tag{4}$$

$$\widehat{W}_{\beta\gamma} \widehat{W}^{R\top\otimes\mathrm{id}}_{\alpha\beta} \widehat{W}^*_{\beta\gamma} = \widehat{W}^{R\top\otimes\mathrm{id}}_{\alpha\beta} \widehat{W}^{R\top\otimes\mathrm{id}}_{\alpha\gamma}, \tag{5}$$

$$W_{\beta\gamma} W_{\alpha\gamma} \widehat{W}_{\alpha\beta} = \widehat{W}_{\alpha\beta} W_{\alpha\gamma}. \tag{6}$$

(2) is just pentagon equations in standard form. To obtain (3) it is enough to apply the algebra homomorphism $\widehat{R}\top$ (the unitary coinverse on $\widehat{A}$ followed by the transposition) to the α leg in (2). Replacing W by the dual $\widehat{W}$ we obtain (4) and (5). We know that $\widehat{W}_{\alpha\beta} = W^*_{\beta\alpha}$. With this information (6) reduces to the pentagon equation in standard form.

Pentagon equation

$$W_{\alpha\gamma}^{\hat{R}\top\otimes\mathrm{id}}\widehat{W}_{\alpha\beta}^{R\top\otimes\mathrm{id}}W_{\beta\gamma}^{*}=\widehat{W}_{\alpha\beta}^{R\top\otimes\mathrm{id}}W_{\alpha\gamma}^{\hat{R}\top\otimes\mathrm{id}}. \qquad (7)$$

To prove (7) we start with the pentagon equation of the form

$$W_{\beta\gamma}^{*}\widehat{W}_{\alpha\beta}W_{\alpha\gamma}=W_{\alpha\gamma}\widehat{W}_{\alpha\beta}$$

Applying to the both sides the algebra antihomomorphism $\top\otimes\hat{R}\otimes R$ ($\top$ acts on α, $\hat{R}$ acts on β and R acts on γ legs) we obtain

$$W_{\alpha\gamma}^{\top\otimes R}\widehat{W}_{\alpha\beta}^{\top\otimes\hat{R}}W_{\beta\gamma}^{*(\hat{R}\otimes R)}=\widehat{W}_{\alpha\beta}^{\top\otimes\hat{R}}W_{\alpha\gamma}^{\top\otimes R}. \qquad (8)$$

We know that $W^{\hat{R}\otimes R}=W$ and $\widehat{W}^{R\otimes\hat{R}}=\widehat{W}$. Inserting in (8), $W^{\hat{R}\otimes R}$ instead of W and $\widehat{W}^{R\otimes\hat{R}}$ instead of $\widehat{W}$ we obtain (7).

Proof

$$\mathbb{W}_{345678} = W_{57} W_{47} \widehat{W}_{58} W_{37}^{\hat{R}\top\otimes\mathrm{id}} \widehat{W}_{38}^{R\top\otimes\mathrm{id}}.$$

$$\mathbb{W}_{345678} \mathbb{W}_{012345} \mathbb{W}^*_{345678} = W_{57} W_{47} \widehat{W}_{58} \mathbb{W}_{012345} \widehat{W}^*_{58} W^*_{47} W^*_{57}$$

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$$W_{47} W_{04}^{\hat{R}\top\otimes\mathrm{id}} W^*_{47} \widehat{W}_{58} \widehat{W}_{05}^{R\top\otimes\mathrm{id}} \widehat{W}^*_{58} W^*_{57}$$

$$= W_{57} W_{24} W_{27} W_{14} W_{17} \widehat{W}_{25} \widehat{W}_{28}$$
$$W_{04}^{\hat{R}\top\otimes\mathrm{id}} W_{07}^{\hat{R}\top\otimes\mathrm{id}} \widehat{W}_{05}^{R\top\otimes\mathrm{id}} \widehat{W}_{08}^{R\top\otimes\mathrm{id}} W^*_{57}$$

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$$= \mathbb{W}_{012345} \mathbb{W}_{012678}.$$

It shows that $\mathbb{W}$ is a multiplicative unitary.

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$$= W_{57} W_{24} W_{27} W_{14} W_{17} \widehat{W}_{25} \widehat{W}_{28} \; W_{04}^{\hat{R}\top\otimes\mathrm{id}} W_{07}^{\hat{R}\top\otimes\mathrm{id}} \widehat{W}_{05}^{R\top\otimes\mathrm{id}} \widehat{W}_{08}^{R\top\otimes\mathrm{id}} W^*_{57}$$

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$$= \mathbb{W}_{012345}\mathbb{W}_{012678}.$$

It shows that $\mathbb{W}$ is a multiplicative unitary.

Proof

$$\mathbb{W}_{345678} = W_{57} W_{47} \widehat{W}_{58} W_{37}^{\hat{R}\top\otimes\mathrm{id}} \widehat{W}_{38}^{R\top\otimes\mathrm{id}}.$$

$$\begin{aligned}
\mathbb{W}_{345678}\mathbb{W}_{012345}\mathbb{W}^*_{345678} &= W_{57} W_{47} \widehat{W}_{58} \mathbb{W}_{012345} \widehat{W}^*_{58} W^*_{47} W^*_{57} \\
&= W_{57} W_{47} W_{24} W^*_{47} W_{47} W_{14} W^*_{47} \widehat{W}_{58} \widehat{W}_{25} \widehat{W}^*_{58} \\
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&= W_{57} W_{24} W_{27} W_{14} W_{17} \widehat{W}_{25} \widehat{W}_{28} \\
&\quad W_{04}^{\hat{R}\top\otimes\mathrm{id}} W_{07}^{\hat{R}\top\otimes\mathrm{id}} \widehat{W}_{05}^{R\top\otimes\mathrm{id}} \widehat{W}_{08}^{R\top\otimes\mathrm{id}} W^*_{57} \\
&= W_{24} \widehat{W}_{25} W_{27} W_{14} W_{17} \widehat{W}_{28} W_{04}^{\hat{R}\top\otimes\mathrm{id}} \widehat{W}_{05}^{R\top\otimes\mathrm{id}} W_{07}^{\hat{R}\top\otimes\mathrm{id}} \widehat{W}_{08}^{R\top\otimes\mathrm{id}} \\
&= \mathbb{W}_{012345}\mathbb{W}_{012678}.
\end{aligned}$$

It shows that $\mathbb{W}$ is a multiplicative unitary.

Manageability of $\mathbb{W}$

Theorem

$\mathbb{W}$ is manageable with

$$\mathbb{Q} = \left(Q^{-1}\right)^{\top} \otimes Q \otimes Q$$

$$\widetilde{\mathbb{W}} = \widehat{W}_{25}^{*(\top\otimes\widehat{R})}\, \widehat{W}_{05}^{*}\, W_{24}^{*(\top\otimes R)}\, W_{14}^{*(\top\otimes R)}\, W_{04}^{*}.$$

Denoting by τ and R the scaling group and the unitary antipode for double, we have

$$\tau_t(a \otimes \widehat{a}) = \tau_t(a) \otimes \widehat{\tau}_t(\widehat{a})$$

$$(a \otimes \widehat{a})^R = \widehat{W}(a^R \otimes \widehat{a}^{\widehat{R}})\widehat{W}^*$$

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Shorthand formula for $\mathbb{W}$.

$$\mathbb{W}_{012345} = W_{\widehat{r}4} \widehat{W}_{r5},$$

where

$$\widehat{r} \in \mathrm{Rep}(\widehat{A}, \overline{H}_0 \otimes H_1 \otimes H_2),$$
$$r \in \mathrm{Rep}(A, \overline{H}_0 \otimes H_2),$$

$$\widehat{r}(\widehat{a}) = \left(\widehat{\Delta}^2(\widehat{a})\right)^{\widehat{R}\top \otimes \mathrm{id} \otimes \mathrm{id}},$$
$$r(a) = \left(\Delta(a)\right)^{R\top \otimes \mathrm{id}}.$$

Commutation formula for r and $\widehat{r}$:

$$(\mathrm{id}^{\otimes 4} \otimes \sigma)\left(W_{\widehat{r}4} \widehat{W}_{r5}\right) = \widehat{W}_{r4} W_{\widehat{r}5}$$

$$W_{45} W_{\widehat{r}5} \widehat{W}_{r4} W^*_{45} = \widehat{W}_{r4} W_{\widehat{r}5}.$$

Alternative formula for $\mathbb{W}$.

Let

$$\mathbb{W}' = (\mathrm{id} \otimes \sigma \otimes \mathrm{id} \otimes \sigma)\mathbb{W}.$$

Then

$$\begin{aligned}\mathbb{W}'_{012345} &= \widehat{W}_{24}\widehat{W}_{14}\widehat{W}_{04}^{R\top\otimes\mathrm{id}}W_{25}W_{05}^{\widehat{R}\top\otimes\mathrm{id}} \\ &= \widehat{W}_{s4}W_{\widehat{s}5},\end{aligned}$$

where

$$s \in \mathrm{Rep}(A, \overline{H}_0 \otimes H_1 \otimes H_2),$$
$$\widehat{s} \in \mathrm{Rep}(\widehat{A}, \overline{H}_0 \otimes H_2),$$

$$s(a) = \left(\Delta^2(a)\right)^{R\top\otimes\mathrm{id}\otimes\mathrm{id}},$$

$$\widehat{s}(\widehat{a}) = \left(\widehat{\Delta}(\widehat{a})\right)^{\widehat{R}\top\otimes\mathrm{id}}.$$

Where the main formula came from

Δ is implemented by W: $\qquad \Delta(a) = W(a \otimes I)W^*$

$\widehat{\Delta}$ is implemented by $\widehat{W}$: $\qquad \widehat{\Delta}(\widehat{a}) = \widehat{W}(\widehat{a} \otimes I)\widehat{W}^*$

$$(\Delta \otimes \widehat{\Delta})(a \otimes \widehat{a}) = W_{12}\widehat{W}_{34}(a \otimes I \otimes \widehat{a} \otimes I)\left(W_{12}\widehat{W}_{34}\right)^*$$

$(\Delta \otimes \widehat{\Delta})$ is implemented by $W_{12}\widehat{W}_{34}$.

$$(\mathrm{id} \otimes \mathrm{flip} \otimes \mathrm{id})(\Delta \otimes \widehat{\Delta})(a \otimes \widehat{a}) = W_{13}\widehat{W}_{24}(a \otimes \widehat{a} \otimes I \otimes I)\left(W_{13}\widehat{W}_{24}\right)^*$$

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Twisted flip

$$\sigma(a \otimes \widehat{a}) = W(\widehat{a} \otimes a)W^*.$$

$$\Delta = (\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\Delta \otimes \widehat{\Delta})$$

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Δ is implemented by $W_{23}W_{13}\widehat{W}_{24}$.

Does $\mathbb{W} = W_{23}W_{13}\widehat{W}_{24}$? If YES then we should have

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(W_{23}W_{13}\widehat{W}_{24}) = W_{23}W_{13}\widehat{W}_{24}W_{25}W_{15}\widehat{W}_{26}$$

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Computation show that

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Let us try $\mathbb{W} = W_{23} W_{13} \widehat{W}_{24} X_{034}$ with X having second leg in A and third in $\widehat{A}$. X_{034} commutes with $I \otimes a \otimes \widehat{a} \otimes I \otimes I$, so it does not spoil the implementation formula. Now

$$(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} \otimes \Delta)\mathbb{W} = \mathbb{W}_{01234} \mathbb{W}_{01256}$$

is equivalent to

$$(\mathrm{id} \otimes \Delta)X = W_{23} X_{012} X_{034}$$

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Where the main formula came from

which in turn is equivalent to

$$(\mathrm{id} \otimes \Delta \otimes \widehat{\Delta})X = X_{013}X_{024}\widehat{W}^*_{23}. \tag{9}$$

Theorem

X is a solution of (9) if and only if

$$X = V_{01}\widehat{V}_{02},$$

where

$$(\mathrm{id} \otimes \Delta)V = V_{01}V_{02},$$
$$(\mathrm{id} \otimes \widehat{\Delta})\widehat{V} = \widehat{V}_{01}\widehat{V}_{02},$$
$$V_{01}\widehat{V}_{02} = \widehat{V}_{02}V_{01}\widehat{W}^*_{12}.$$

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Theorem

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$$V_{01} \widehat{V}_{02} = \widehat{V}_{02} V_{01} \widehat{W}^*_{12}.$$

Where the main formula came from

Compare

$$\begin{aligned}(\mathrm{id}\otimes\Delta)V &= V_{01}V_{02},\\ (\mathrm{id}\otimes\widehat{\Delta})\widehat{V} &= \widehat{V}_{01}\widehat{V}_{02},\\ V_{01}\widehat{V}_{02} &= \widehat{V}_{02}V_{01}\widehat{W}^{*}_{12},\end{aligned}$$

$$\begin{aligned}(\mathrm{id}\otimes\Delta)W &= W_{01}W_{02},\\ (\mathrm{id}\otimes\widehat{\Delta})\widehat{W} &= \widehat{W}_{01}\widehat{W}_{02},\\ \widehat{W}_{02}W_{01} &= \widehat{W}^{*}_{12}W_{01}\widehat{W}_{02}.\end{aligned}$$

Solution:

$$V = W^{\top\otimes R} = W^{\widehat{R}\top\otimes\mathrm{id}},$$

$$\widehat{V} = \widehat{W}^{\top\otimes\widehat{R}} = \widehat{W}^{R\top\otimes\mathrm{id}},$$

$$X = W_{01}^{\widehat{R}\top\otimes\mathrm{id}}\widehat{W}_{02}^{R\top\otimes\mathrm{id}}$$

and formula (1) follows.

Where the main formula came from

Compare

$$
\begin{aligned}
(\mathrm{id}\otimes\Delta)V &= V_{01}V_{02},\\
(\mathrm{id}\otimes\widehat{\Delta})\widehat{V} &= \widehat{V}_{01}\widehat{V}_{02},\\
V_{01}\widehat{V}_{02} &= \widehat{V}_{02}V_{01}\widehat{W}^{*}_{12},
\end{aligned}
\qquad
\begin{aligned}
(\mathrm{id}\otimes\Delta)W &= W_{01}W_{02},\\
(\mathrm{id}\otimes\widehat{\Delta})\widehat{W} &= \widehat{W}_{01}\widehat{W}_{02},\\
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\end{aligned}
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\end{aligned}
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and formula (1) follows.

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$$\begin{aligned}(\mathrm{id}\otimes\Delta)W &= W_{01}W_{02},\\ (\mathrm{id}\otimes\widehat{\Delta})\widehat{W} &= \widehat{W}_{01}\widehat{W}_{02},\\ \widehat{W}_{02}W_{01} &= \widehat{W}^*_{12}W_{01}\widehat{W}_{02}.\end{aligned}$$

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$$X = W_{01}^{\widehat{R}\top\otimes\mathrm{id}}\widehat{W}_{02}^{R\top\otimes\mathrm{id}}$$

and formula (1) follows.

Example

For the first time, double group construction was used in 1990 to construct a quantum deformation of Lorentz group. With some abuse of terminology by Lorentz group we mean $SL(2, \mathbb{C})$ considered as **real** Lie group.

Quantum Lorentz group.

$0 < q < 1$. Quantum Lorentz group is a quantum matrix group. The algebra A is generated by matrix elements of $S_qL(2,\mathbb{C})$-matrix:

$$u = \begin{pmatrix} \alpha \ , & \beta \\ \gamma \ , & \delta \end{pmatrix}$$

The comultiplication acts on generators in the following way:

$$(\mathrm{id} \otimes \Delta)u = u_{12}u_{13}.$$

Explicitly

$$\begin{pmatrix} \Delta(\alpha) \ , & \Delta(\beta) \\ \Delta(\gamma) \ , & \Delta(\delta) \end{pmatrix} = \begin{pmatrix} \alpha\otimes\alpha + \beta\otimes\gamma \ , & \alpha\otimes\beta + \beta\otimes\delta \\ \gamma\otimes\alpha + \delta\otimes\gamma \ , & \gamma\otimes\beta + \delta\otimes\delta \end{pmatrix}.$$

Does Δ exist? Show that RHS is a $S_qL(2,\mathbb{C})$-matrix!

$S_qL(2,\mathbb{C})$ - commutation relations

$u = \begin{pmatrix} \alpha , & \beta \\ \gamma , & \delta \end{pmatrix}$ is an $S_qL(2,\mathbb{C})$-matrix if

$$\begin{aligned}
\alpha\beta &= q\beta\alpha, & \gamma\alpha^* &= q\alpha^*\gamma, \\
\alpha\gamma &= q\gamma\alpha, & \delta\alpha^* &= \alpha^*\delta, \\
\alpha\delta - q\beta\gamma &= I, & \gamma\beta^* &= \beta^*\gamma, \\
\beta\gamma &= \gamma\beta, & \delta\gamma^* &= q^{-1}\gamma^*\delta, \\
\beta\delta &= q\delta\beta, & \alpha\alpha^* &= \alpha^*\alpha + (1-q^2)\gamma^*\gamma, \\
\gamma\delta &= q\delta\gamma, & \gamma\gamma^* &= \gamma^*\gamma, \\
\delta\alpha - q^{-1}\beta\gamma &= I, & \delta\delta^* &= \delta^*\delta - (1-q^2)\gamma^*\gamma,
\end{aligned}$$

$$\begin{aligned}
\beta\alpha^* &= q^{-1}\alpha^*\beta + q^{-1}(1-q^2)\gamma^*\beta, \\
\delta\beta^* &= q\beta^*\delta - q(1-q^2)\alpha^*\gamma, \\
\beta\beta^* &= \beta^*\beta + (1-q^2)(\delta^*\delta - \alpha^*\alpha) - (1-q^2)^2\gamma^*\gamma.
\end{aligned}$$

These are 17 relations of Podleś (1989).

$S_qU(2)$ - commutation relations

$u = \begin{pmatrix} \alpha \ , & \beta \\ \gamma \ , & \delta \end{pmatrix}$ is an $S_qU(2)$-matrix if u is a unitary $S_qL(2,\mathbb{C})$-matrix.

Then $\beta = q\gamma^*$, $\delta = \alpha^*$ and

$$\begin{aligned} \alpha\gamma &= q\gamma\alpha, \\ \alpha\gamma^* &= q\gamma^*\alpha, \\ \gamma\gamma^* &= \gamma^*\gamma, \end{aligned} \qquad \begin{aligned} \alpha^*\alpha + \gamma^*\gamma &= I, \\ \alpha\alpha^* + q^2\gamma^*\gamma &= I. \end{aligned}$$

These are 5 relations of quantum $SU(2)$.
Let A be the algebra generated by matrix elements of $S_qU(2)$-matrix. Then there exists $\Delta \in \mathrm{Mor}(A, A \otimes A)$ such that

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$\widehat{S_qU(2)}$ - commutation relations

$u = \begin{pmatrix} \alpha , & \beta \\ \gamma , & \delta \end{pmatrix}$ is an $\widehat{S_qU(2)}$-matrix if u is an upper triangular $S_qL(2,\mathbb{C})$-matrix with positive selfadjoint elements on the diagonal.

Then $\gamma = 0$, $\alpha^* = \alpha$, $\delta = \alpha^{-1}$ and

$$\alpha\beta = q\beta\alpha,$$
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These are 2 relations of quantum $\widehat{SU(2)}$.
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Iwasawa decomposition

Theorem

Let u be an $S_qL(2,\mathbb{C})$-matrix. Then there exist unique $S_qU(2)$-matrix u and $\widehat{S_qU(2)}$-matrix u such that

$$u = uu.$$

Moreover the matrix elements of u commutes with matrix elements of u.

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Comultiplication for $S_qL(2,\mathbb{C})$

We want $\Delta \in \mathrm{Mor}(A, A\otimes A)$ such that

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We need $\sigma \in \mathrm{Mor}(A\otimes A, A\otimes A)$ such that

$$(\mathrm{id}\otimes\sigma)(u_{13}u_{14}) = u_{13}u_{14}.$$

Solution $\quad \sigma(a\otimes\hat{a}) = W(\hat{a}\otimes a)W^*.$

Then

$$\Delta = (\mathrm{id}\otimes\sigma\otimes\mathrm{id})\circ(\Delta\otimes\Delta).$$

Hence Quantum Lorentz Group is the result of **Double Group Construction** applied to Quantum $SU(2)$.

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