Subfactors in relation with quantum groups

▸ Référence

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Von Neumann algebras

Definition

A von Neumann algebra M is any weakly closed *-unitary subalgebra of $\mathcal{L}(H)$ (H : Hilbert space)

- if $X \subset \mathcal{L}(H)$ set $X' = \{a \in \mathcal{L}(H) / \forall x \in X, xa = ax\}$

Proposition

If M is a unitary *-subalgebra of $\mathcal{L}(H)$, M VN algebra iff $M'' = M$

Examples of von Neumann algebras

Commutative VN alg : $L^\infty(X,\mu)$ $(H = L^2(X,\mu))$

If G is a loc compact group with Haar measure, if $s \in G$, $f \in L^2(G)$
$\lambda(s)f(t) = f(s^{-1}t)$ the $\lambda(s)$'s generates $\mathcal{L}(G) \subset \mathcal{L}(L^2(G))$

$$\mathcal{L}(G)' = \mathcal{R}(G)$$

Definition

A Factor is any VN algebra M such that $Z(M) = M \cap M' = \mathbb{C}1$

$\mathcal{L}(G)$ factor iff G ICC

Classification of factors

$M = \int M(x)dx$ where $M(x)$ is a factor
$(36')$ (Murray von Neumann) $\rightarrow$ 75' (Connes)
Classification of separable hyperfinite factors $(M = \overline{\bigcup_n M_n})$ $\dim M_n$ *finite*)

Type	*VNalg*
I_n	$M_n(\mathbb{C})$
I_∞	$\mathcal{L}(H)$
II_1	R (*Connes*)
II_∞	$R\overline{\otimes}\mathcal{L}(H)$ (*Connes*)
III	.....

II_1 : $\exists\tau : M \rightarrow \mathbb{C}$ lin.form $\tau(x^*x) > 0$ if $x \neq 0$, trace: $\tau(xy) = \tau(yx)$
II_∞: τ is a weight ("unbounded")

$$R = "\overline{\bigotimes_{n\in\mathbb{N}} M_n(\mathbb{C})}", \ \tau = "\overline{\bigotimes_{n\in\mathbb{N}} \tau_n}"$$

Inclusions of VN alg.

$$M_0 \subset M_1 \quad \text{to be classified}$$

Basic construction (Jones 83')

$$\begin{array}{ccccccccc}
M_0 & \subset & M_1 & \subset & M_2 & \subset & M_3 & \subset \ldots\ldots \\
 & & \cup & & \cup & & \cup & \\
 & & M_0' \cap M_1 & \subset & M_0' \cap M_2 & \subset & M_0' \cap M_3 & \subset \ldots\ldots \\
 & & & & \cup & & \cup & \\
 & & & & M_1' \cap M_2 & \subset & M_1' \cap M_3 & \subset \ldots\ldots \\
 & & & & & & \cup & \\
 & & & & & & M_2' \cap M_3 & \subset \ldots\ldots
\end{array}$$

Depth 2 : $M_0' \cap M_1 \subset M_0' \cap M_2 \subset M_0' \cap M_3 \subset \ldots\ldots$ basic

Irreducible: $M_0' \cap M_1 = \mathbb{C}$

Index for subfactors

$M_0 \subset M_1$, type II_1-subfactors of $\mathcal{L}(H)$

Index: $[M_1 : M_0] = \frac{\dim_{M_0}(H)}{\dim_{M_1}(H)}$

Theorem Jones 83'

$\{[M_1 : M_0]/M_i \ \ type\ II_1\} = \{4cos^2\frac{\pi}{n}/n = 3....\} \cup [4, \infty]$

QUESTION: Is any number in $[4, \infty]$ obtained for an irreducible inclusion?

Actions of groups on VN alg.

Action of G on M: continous homomorphism $\alpha : G \to Aut(M)$

Examples

- If G acts on (X, μ), it acts on $L^\infty(X, \mu)$: $\alpha_g(f)(t) = f(t \triangleleft g)$
- If G is finite ($G \subset S_n \subset S_\infty(fini.support)$) it acts on $R = \overline{\bigotimes_{i \in S_\infty} M_2(\mathbb{C})}$ by

  Bernoulli shifts: $\alpha_g((a_f)_{f \in S_\infty}) = (a_{g \circ f})_{f \in S_\infty}$

Associated VN alg

- $M^G = \{m / \forall g \alpha_g(m) = m\}$ if $H < G$: $[M^H : M^G] = [G : H]$
- $M \rtimes G \subset \mathcal{L}(H \otimes L^2(G)))(= \mathcal{L}(L^2(G, H))$:

$$< (g \mapsto \alpha_g(m))_{m \in M}, 1 \otimes \mathcal{R}(G) >$$

$$\alpha(M^G) \subset \alpha(M) \subset M \rtimes G$$

Quantum groups in VN alg. framework

Definition Kustermans Vaes 99'

(M, Γ, ϕ, ψ) quantum group:

- $\Gamma : M \to M \overline{\otimes} M \quad (\Gamma \otimes i)\Gamma = (i \otimes \Gamma)\Gamma$
- ϕ, ψ n.sf.f. weights on M $\forall x > 0$ in M:
  $(\phi \otimes i)\Gamma(x) = \phi(x)1$ and $(i \otimes \psi)\Gamma(x) = \psi(x)1$

Examples:

- $\dim M < \infty$: Kac algebras $(M, \Gamma, \kappa, \epsilon)$
- Commutative case: $M = L^\infty(G)$, $\Gamma(f)(x, y) = f(xy)$
  $\phi(f) = \int_G f(s)ds$, $\psi(f) = \int_G f(s)d's$
- Symmetric case ($\varsigma\Gamma = \Gamma$): $M = \mathcal{R}(G)$, $\Gamma(\rho(s)) = \rho(s) \otimes \rho(s)$
  $\phi(\rho(f)) = \psi(\rho(f)) = f(e)$

Actions of Quantum groups on VN alg.

Definition

Action of (M, Γ, ϕ, ψ) on A (VN alg): $\alpha : A \to A \otimes M$

$$(\alpha \otimes i)\alpha = (i \otimes \Gamma)\alpha$$

Examples:

- M acts on itself using Γ
- Commutative case: $M = L^\infty(G)$, $L^\infty(G) \otimes M = L^\infty(G, M)$: there exists an action $g \mapsto \alpha_g$ such that: $\alpha(m)(g) = \alpha_g(m)$

Assoc.VN alg.: $M^\alpha = \{m \in M / \alpha(m) = m \otimes 1\}$, $A \rtimes M$

A theorem and its previous reciproque !

Theorem Vaes 01' and 05'

$\alpha(M) \subset A \rtimes M$ is always depth 2. It exists a type III factor A on which any quantum group M acts with $\alpha(M) \subset A \rtimes M$ irreducible (strictly outer), but not any quantum group can act stric. out. on any factor.

Theorem Enock Nest 96'

- Let $M_0 \subset M_1$ a regular depth 2 irreducible inclusion ($M_0' \cap M_1 = \mathbb{C}$), then $M_0' \cap M_2$ and $M_1' \cap M_3$ have can. struct. of quantum group.
- There exists an action $\mathbf{a}$ of $M_1' \cap M_3$ on M_1, such that:
  $M_0 \subset M_1 \subset M_2 \quad \equiv \quad a(M_1^{\mathbf{a}}) \subset a(M_1) \subset M_1 \rtimes_{\mathbf{a}} (M_1' \cap M_3)$

The finite index case

Theorem Szymanski 94'

Référence

$M_0 \subset M_1$: II_1 factors (τ), irreducible, depth 2, finite index $\lambda = [M_1 : M_0]$

$M_0 \subset M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3$..... basic construction, then:

- $A = M_0' \cap M_2$, $B = M_1' \cap M_3$: finit. Kac algebras in duality
  there exists an action $\mathbf{a}$ of B on M_1, such that $M_0 = M_1^{\mathbf{a}}$, and $M_2 = M_1 \rtimes_{\mathbf{a}} B$
- $[M_1 : M_0] = \dim B$

Canonical bracket: $\boxed{< a, b >= \lambda^{-2}\tau(ae_2e_1b).}$ $< \Gamma(a), b \otimes c >=< a, bc >$

$< \kappa(a), b >= \overline{< a^*, b^* >}$ $\quad \epsilon(a) =< a, 1 >$ YOU CAN COMPUTE!!!!!!

A family of examples

Definition

A matched pair H, K of subgroups of a finite group G is any pair of subgroups such that: $G = HK = \{hk/h \in H, k \in K\}$ and $H \cap K = \{e\}$

Let us consider an outer and proper action of G on R: $\alpha : G \to AutR$
if $\mathrm{g} \neq \mathrm{e}$ then $\alpha_{\mathrm{g}} \notin \mathrm{IntR}$

Theorem

If H, K matched pair of finite groups and $G = HK$ acts properly and outerly on the hyperfinite type II_1 factor R, then:

- $M_0 = R^H \subset M_1 = R \rtimes K$ is in the conditions of Sym th.
- The quantum groupoids $M_0' \cap M_2$ and $M_1' \cap M_3$ can be expressed with a double groupoid and some crossed products.

Quantum groups associated with matched pairs

1) The double groupoids $\mathcal{T}$, $\mathcal{T}'$. Référence

$$\mathcal{T} = \{ k' \begin{array}{c} h \\ \boxed{} \\ h' \end{array} k / \;\; h, h' \in H, k, k' \in K \;\; hk = k'h' \}$$

$\mathcal{T} \to \mathcal{T}'$: transposition

$$(k' \begin{array}{c} h \\ \boxed{} \\ h' \end{array} k)^t = h \begin{array}{c} k' \\ \boxed{} \\ k \end{array} h'$$

$\mathcal{T}$ is a groupoid with basis K and a groupoid with basis H

- Horizontal product: $\begin{array}{c} a \\ c\,\square\,\mathbf{k} \\ d \end{array} \overset{D}{\underset{K}{\star}} \begin{array}{c} a' \\ \mathbf{k}\,\square\,b' \\ d' \end{array} = \begin{array}{c} aa' \\ c\,\square\,b' \\ dd' \end{array}$
- Vertical product

$$\begin{array}{c} a \\ c\,\square\,b \\ \mathbf{h} \end{array} \overset{B}{\underset{H}{\star}} \begin{array}{c} \mathbf{h} \\ c'\,\square\,b' \\ d' \end{array} = \begin{array}{c} a \\ cc'\,\square\,bb' \\ d' \end{array}$$

- Horizontal unities : $\begin{array}{c} e \\ k\,\square\,k \\ e \end{array}$, and vertical $\begin{array}{c} h \\ e\,\square\,e \\ h \end{array}$
- Inverse: $\left(\begin{array}{c} a \\ c\,\square\,b \\ d \end{array}\right)^{-D} = \begin{array}{c} a^{-1} \\ b\,\square\,c \\ d^{-1} \end{array}$, $\left(\begin{array}{c} a \\ c\,\square\,b \\ d \end{array}\right)^{-B} = \begin{array}{c} d \\ c^{-1}\,\square\,b^{-1} \\ a \end{array}$

The VN algebras $\mathbb{C}\mathcal{T}$ and $\mathbb{C}\mathcal{T}'$

- Product: $(\sum_{t\in\mathcal{T}} a_t t).(\sum_{t'\in\mathcal{T}} b_{t'} t') = \sum_{t\in\mathcal{T}} \sum_{t_1 \overset{D}{\underset{K}{\star}} t_2 = t} (a_{t_1} b_{t_2}) t$
- Involution: $(\sum_{t\in\mathcal{T}} a_t t)^* = (\sum_{t\in\mathcal{T}} \overline{a_t} t^{-D})$

$\mathbb{C}\mathcal{T}$ and $\mathbb{C}\mathcal{T}'$ are crossed products:

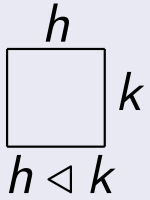
$$g = kh,\ h, k \text{ unique},\ g = p_1(g)p_2(g)$$

Mutual actions of H and K

$$h \triangleright k = p_1(hk), \quad h \triangleleft k = p_2(hk): \quad hk = (h \triangleright k)(h \triangleleft k)$$

$C(K) \rtimes H$ generated by $V_h \chi_k$.

Proposition:

$V_h \chi_k \mapsto$ $h \triangleright k$ 

k : isomorphism between $C(K) \rtimes H$ and $\mathbb{C}\mathcal{T}$

Quantum groups associated with matched pairs

Theorem [Andruskiewicz-Natale] Référence

$\mathbb{C}\mathcal{T}$ has a quantum group structure:

- $\Gamma(t) = \sum_{t_1 \overset{B}{\underset{H}{\star}} t_2 = t} t_1 \otimes t_2 = \sum_{t_1 \overset{B}{\underset{H}{\star}} t_2 = t} t_1 \otimes t_2$
- $\kappa(t) = t^{-DB}$
- $\epsilon(t) = \begin{cases} 1 & \text{if } t \text{ of the form } e \begin{array}{c} h \\ \square \\ h \end{array} e \\ 0 & \text{otherwise} \end{cases}$

The inclusion $R^H \subset R \rtimes K$ (matched pairs)

Theorem Szymanski JMV

H, K matched pairs, $G = HK$ acts outerly and properly on R,

- $M_0 = R^H \subset M_1 = R \rtimes K$ is in NV th conditions
- There exists an isomorphism $\theta_I : M_0' \cap M_2 \to \mathbb{C}\mathcal{T}$
- There exists an isomorphism $\theta_J : M_1' \cap M_3 \to \mathbb{C}\mathcal{T}'$
- The dualities are preserved: , $\forall x \in \mathcal{T}, x' \in \mathcal{T}' :< \theta_I^{-1}(x), \theta_J^{-1}(x') >= \delta_{x',x^t}$

NV

Quantum groupoid: $(A, \Gamma, \kappa, \epsilon)$?

Définition

- A : finite dim. VN algebra ($A = \bigoplus_{i=1,..k} M_{n_i}(\mathbb{C})$)
- $\Gamma : A \to A \otimes A$: $*$-homom.
  $(\Gamma \otimes i)\Gamma = (i \otimes \Gamma)\Gamma$ ($\Gamma(1) \neq 1$, en gén.)
- $\kappa : A \to A$, linear antimultiplicative :
  $(\kappa \circ *)^2 = i$
  $\varsigma(\kappa \otimes \kappa)\Gamma = \Gamma \circ \kappa$
  $(m(\kappa \otimes i) \otimes i)(\Gamma \otimes i)\Gamma(x) = (1 \otimes x)\Gamma(1)$
  $\qquad (\mathrm{where\ m(a \otimes b) = ab})$
- $\epsilon : A \to \mathbb{C}$: linear
  $(\epsilon \otimes i)\Gamma = (i \otimes \epsilon)\Gamma = i$
  $(\epsilon \otimes \epsilon)((x \otimes 1)\Gamma(1)(1 \otimes y)) = \epsilon(xy)$
- $A_t := \{a \in A / \Gamma(a) = (a \otimes 1)\Gamma(1) = \Gamma(1)(a \otimes 1)\}$
  $A_s := \{a \in A / \Gamma(a) = (1 \otimes a)\Gamma(1) = \Gamma(1)(1 \otimes a)\}$

Commutative quantum groupoid

$\mathcal{G}$ **finite groupoid**

- $A = L^\infty(\mathcal{G}) \ \ (\subset \mathcal{L}(l^2(\mathcal{G}))), \quad A \otimes A = L^\infty(\mathcal{G} x \mathcal{G})$
- $\Gamma(f)(x,y) = \begin{cases} f(xy) & \text{x, y composables} \\ 0 & \text{otherwise} \end{cases}$
- $\kappa(f)(x) = f(x^{-1})$
- $\epsilon(f) = \sum_{u \in \mathcal{G}^0} f(u)$
- $A_t = \{\phi \circ r / \phi \in L^\infty(G^0)\}, A_s = \{\phi \circ s / \phi \in L^\infty(G^0)\}$

Symmetric example

$\mathcal{G}$ **finite groupoid**

- $\hat{A} = \mathcal{L}(\mathcal{G})(= \mathbb{C}(\mathcal{G})) \quad (\subset \mathcal{L}(l^2(\mathcal{G})))$
- $\hat{\Gamma}(\lambda(s)) = \lambda(s) \otimes \lambda(s)$
- $\hat{\kappa}(\lambda(s)) = \lambda(s^{-1})$
- $\hat{\epsilon}(\lambda(s)) = 1$
- $\hat{A}_t = \hat{A}_s = A_t$

Quantum groupoid action

Definition

Right action of $(A, \Gamma, \kappa, \epsilon)$ on a VN alg M : (b, α)

- $b : A_t \hookrightarrow M$: 1 to 1, unital *-antihom.
- $\alpha : M \hookrightarrow M \otimes A$, 1 to 1 *-hom. $(\alpha(1) \neq 1)$
  $\forall n \in A_t : \alpha(1)(b(n) \otimes 1) = \alpha(1)(1 \otimes n) \qquad (\alpha(M) \subset M_b \underset{A_t}{\star}{}_i A)$
- $(\alpha \otimes i)\alpha = (i \otimes \Gamma)\alpha$
- $\forall n \in A_t : \qquad \alpha(b(n)) = \alpha(1)(1 \otimes \kappa(n))$

Crossed product

$M \ltimes A :=< \alpha(M) \cup \alpha(1)(1 \otimes \hat{A}') > \big(\subset (M \otimes \mathcal{L}(H_A))_{\alpha(1)} \big)$

Fixed point algebra

$M^A := \{m \in M / \alpha(m) = \alpha(1)(m \otimes 1)\}$

$$M^A \subset M \ltimes A$$

Quantum groupoids and II_1 subfactors

Theorem Nikshych-Vainerman 00',David 05'

Référence

$M_0 \subset M_1$: II_1 factors (τ), depth 2, finite index $\lambda = [M_1 : M_0]$

$M_0 \subset M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} M_3$..... basic construction, then:

- $A = M_0' \cap M_2$, $B = M_1' \cap M_3$: quantum groupoids, $M_0' \cap M_1$ common basis, it exists $\mathbf{a}$ act. of B on M_1 with $M_0 = M_1^{\mathbf{a}}$, and $M_2 = M_1 \rtimes_{\mathbf{a}} B$
- $[M_1 : M_0] = [B : B_t]$
- there exists a natural Galois correspondance between the lattice of intermediate subfactors $M_1 \subset P \subset M_2$ and the lattice of left coideal *-subalgebras of $M_1' \cap M_2$

Bracket : $\boxed{< a, b >= \lambda^{-2}\tau(ahe_2e_1hb).}$ $< \Gamma(a), b \otimes c >=< a, bc >$

$< \kappa(a), b >= \overline{< a^*, b^* >}$ $\epsilon(a) =< a, 1 >$ YOU CAN COMPUTE

Quantum groupoids and II_1 subfactors

Theorem Nikshych-Vainerman,David

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Adapted pairs of groups

Définition

H, K finite subgroups of G, **adapted pair** if:

$$G = HK = \{hk / h \in H, k \in K\}$$

Remark: $g = hk = hxx^{-1}k \qquad x \in H \cap K$

One can define the double groupoids $\mathcal{T}$ and $\mathcal{T}'$, the quantum groupoids $(\mathbb{C}\mathcal{T}, \mathbb{C}\mathcal{T}')$, they still are crossed products and finally:

The inclusion $R^H \subset R \rtimes K$ (adapted pairs of groups)

Theorem JMV 08'

If H, K adapted pair of groups, and let $G = HK$ act outerly and properly on R,

- $M_0 = R^H \subset M_1 = R \rtimes K$ is in NV theorem conditions
- It exists an isomorphism $\theta_I : M_0' \cap M_2 \to \mathbb{C}\mathcal{T}$
- It exists an isomorphism $\theta_J : M_1' \cap M_3 \to \mathbb{C}\mathcal{T}'$
- The dualities are preserved: ,
  $\forall x \in \mathcal{T}, x' \in \mathcal{T}' :< \theta_I^{-1}(x), \theta_J^{-1}(x') >= |H \cap K|.\delta_{x',x^t}$

NV

Remarks on the index

Remarks

- $[R \rtimes K : R^H] = |H||K| \in \mathbb{N}$
- Quantum groupoids actions lead to algebraic integers
- In order to obtain other values of the index (non irreducible case), one can use a reconstruction theorem which gives quantum groupoids from fusion categories (L. Vainerman's talks)
  Example: the Tambara Yamagami categories give all numbers of the form $(n + \sqrt{n})^2$ (C.Mevel's Thesis 2010 in Caen)
- You need (not finite) quantum groups actions on not hyperfinite factors to answer exactly Jones question (see tomorrow Fima's talk).....

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