7th focused semester on Quantum Groups

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1. Quantum groups and applications
   - Quantum groups
   - Subfactors
   - Universal and free quantum groups
   - Noncommutative Geometry and K-theory

2. Organization of the special semester
   - Graduate courses
   - Workshops
   - Conference

Quantum groups

Idea : encode the group structure in an algebra A and a coproduct $\Delta : A \to A \otimes A$ (and maybe also an antipode...)
G compact group ➔ $A = C(G)$ with coproduct $\Delta(\varphi)(g,h) = \varphi(gh)$
Γ discrete group ➔ $\mathcal{A} = \mathbb{C}\Gamma$ with coproduct $\Delta(\gamma) = \gamma \otimes \gamma$ for $\gamma \in \Gamma$
$\mathfrak{G}$ Lie algebra ➔ $U\mathfrak{G}$ with coproduct $\Delta(X) = X \otimes 1 + 1 \otimes X$ for $X \in \mathfrak{G}$

One motivation : Pontrjagin duality for non abelian groups.
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Let quantum groups act ! $G \curvearrowright X$ yields $\delta_X : C(X) \to C(G \times X) \simeq$
$\simeq C(G) \otimes C(X)$ given by $\delta(\varphi)(g,x) = \varphi(g \cdot x)$.
Quantum action : coaction $\delta_B : B \to A \otimes B$ of (A, Δ) on another algebra B. One can construct a crossed product $B \rtimes A$ with coaction of $\hat{A}$.
Baaj-Skandalis 1993 : general framework for Takesaki-Takai duality. In "good cases", $B \rtimes A \rtimes \hat{A}$ is covariantly stably isomorphic to B.

In the 1980's, new series of examples coming from physical motivations (Yang-Baxter equation...)

Drinfeld-Jimbo 1985 : q-deformations $U_q\mathfrak{G}$ for $\mathfrak{G}$ complex simple
Woronowicz 1987 : $SU_q(n)$, general definition of compact quantum groups
Rosso 1988 : the restricted duals $(U_q\mathfrak{G})^\circ$ fit into Woronowicz' framework
Kustermans-Vaes 2000 : locally compact quantum groups

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Key examples :

- $SU_q(2)$, "non unimodular" compact quantum group
- "Quantum $ax + b$" groups, with scaling constant $\nu \neq 1$ (Woronowicz 1999)
- Cocycle bicrossed products arising from "matched pairs" $G_1 G_2 \subset G$, yielding non-semi-regular l.c. quantum groups (Majid, Baaj, Skandalis, Vaes 1991–2003)

Subfactors

Factor : von Neumann algebra M such that $Z(M) = M' \cap M = \mathbb{C}1$.

Consider an inclusion of factors $M_0 \subset M_1$ and the associated Jones' tower $M_0 \subset M_1 \subset M_2 \subset M_3 \subset \cdots$. Assume the inclusion is irreducible ($M_0' \cap M_1 = \mathbb{C}1$), regular, and has depth 2 ($M_0' \cap M_3$ is a factor).
Example : $M^G \subset M \subset M \rtimes G \subset \cdots$ for every outer integrable action of a l.c. group G on a factor M.

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Ocneanu, Enock-Nest 1996 : all such inclusions are of the above form with G a locally compact *quantum* group, given by $L^\infty(G) = M_0' \cap M_2$ and $L^\infty(\hat{G}) = M_1' \cap M_3$.

Vaes 2005 : not every l.c. compact quantum group can act outerly on any factor (obstruction related to Connes' T invariant). There exist a type III_1 factor on which every l.c. quantum group can act outerly.

Jones 1983 : does the hyperfinite II_1 factor R admits *irreducible* subfactors of any index $\lambda > 4$?

Wasserman inclusions : if G acts outerly on N and π is an irreducible representation of G, consider $M_0 = 1 \otimes N^G \subset (B(H_\pi) \otimes N)^G = M_1$. This is an irreducible inclusion with index $(\dim \pi)^2$ relatively to the natural condition expectation $E : M_1 \to M_0$.

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With quantum groups, $\dim \pi$ can take non-integer values! But the factors can be type III... Take $G = SU_q(2)$ and π its fundamental representation, so that $\dim \pi = q + q^{-1}$.
Take $N = L(F_n) * L^\infty(G)$ with trivial action on the first factor. Let $\phi = \tau * h$ be the free product state on N.

Shlyakhtenko-Ueda 2001 : The inclusion of the centralizers $M_0^\phi \subset M_1^{\phi E}$ is an inclusion of type II_1 factors with the same index and relative commutants as $M_0 \subset M_1$.
$L(F_\infty)$ admits irreducible subfactors of any index $\lambda > 4$.

Liberation of quantum groups

The fonction algebras $C(U_n)$, $C(O_n)$, $C(S_n)$ can be described by generators and relations as follows.
Calling u_{ij}, $1 \leq i, j \leq n$ the generators and putting $U = (u_{ij})_{ij}$ we have

- $C(U_n) = \langle 1, u_{ij} \mid [u_{ij}, u_{kl}] = 0, UU^* = U^*U = I_n \rangle_{C^*}$
- $C(O_n) = \langle 1, u_{ij} \mid [u_{ij}, u_{kl}] = 0, U = U^*, UU^* = U^*U = I_n \rangle_{C^*}$
- $C(S_n) = \langle 1, u_{ij} \mid [u_{ij}, u_{kl}] = 0, u_{ij}^2 = u_{ij}, \sum_k u_{ik} = \sum_k u_{kj} = 1 \rangle_{C^*}$

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Remove the vanishing of commutators → C^*-algebras $A_u(n)$, $A_o(n)$, $A_s(n)$.
Coproduct $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ → "free" compact quantum groups.

Wang 1998 : the "quantum group of permutations of 4 points" is infinite, i.e. $\dim A_s(4) = +\infty$.

Banica 1996–1998 : representation theory of "liberated quantum groups".

- $A_o(n)$ has the same fusion rules as $SU(2)$
- $A_s(n)$ has the same fusion rules as $SO(3)$
- $A_u(n)$ has irreducibles coreps indexed by words in u, $\bar{u}$ with recursive fusion rules : $vu \otimes uw = wuuw$, $v\bar{u} \otimes uw = v\bar{u}uw \oplus v \otimes w$, ...

Dual point of view : compare $A_*(n)$ with group C^*-algebras $C^*(\Gamma)$.
We have e.g. a regular representation $A \to B(H)$ (Haar state) and a trivial representation $A \to \mathbb{C}$ (co-unit).
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Vaes-Vergnioux 2007 : like free group factors, the von Neumann algebras $A_o(n)''$ are full and prime II_1 factors.
Vergnioux 2010 : unlike the one of free groups, the first L^2-Betti number $\beta_1^{(2)}(\mathcal{A}_o(n))$ vanishes.

These results use methods inspired from geometric group theory.

Noncommutative Geometry

The classical sphere S^3 is the quotient of $SU(2)$ by its maximal torus T.
The q-deformation $SU_q(2)$ still "contains" T $\rightarrow$ Podleś' sphere S^3_q :

- given by a noncommutative C^*-algebra $C(S^3_q)$
- naturally equipped with an action of $SU_q(2)$
- and with a Dirac operator D on a "natural" $C(S^3_q)$-module

D is defined diagonnally on quantum "spherical harmonics" coming from the knowledge of the representation theory of $SU_q(2)$.

Nest-Voigt 2009 : As in the classical case, using D one proves that $C(S^3_q)$ is $KK^{SU_q(2)}$-equivalent to $\mathbb{C} \oplus \mathbb{C}$.

Special feature of the quantum case : additional symmetry.
In fact the Drinfeld double $D(SU_q(2))$ acts on S_q^3 !

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Using work of Meyer, Nest, Vaes, this implies the Baum-Connes conjecture for the dual of $SU_q(2)$, and by monoidal equivalence, for the duals of the universal orthogonal quantum groups $A_o(Q)$.
Consequence : the full and reduced versions of $A_o(Q)$ have K_0 and K_1 groups equal to $\mathbb{Z}$.

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Question : what about the "quantum space" $SU_q(2)$ itself ?
Spectral triples have been constructed and studied by Chakraborty-Pal, Connes, Dabrowski-Landi et al., but they do not satisfy the strongest requirements of noncommutative geometry...

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Graduate courses

Lectures series :

- B. Leclerc (ICM 2010 lecturer) *Introduction to quantum enveloping algebras*
- R. Vergnioux *Compact and discrete quantum groups*
- L. Vainerman *Representations of quantum groups and applications to subfactors and topological invariants*

Mini-lectures :

- T. Banica *Quantum permutation groups*
- Ch. Voigt *Quantum groups and NCG*
- S. Sundar *Odd dimensional quantum spheres*

Quantum Groups and physics

Date : 6- 10 September 2010
Location : Caen

Mini-lectures by John Barrett :

- $U_q(sl(2))$ at root of unity
- Turaev-Viro topological quantum field theory for 3-manifolds

List of invited talks : Arzano, Girelli, Kasprzak, Kowalski, Lukierski, Majid, Martinetti, Meusburger, Nikshych, Noui, Perez, Tolstoiy, ...

GREFI-GENCO meeting

Date : 27 September 1 October 2010
Location : Marseille (CIRM)

Mini-lectures :

- B. Collins *Weingarten calculus and applications to quantum groups*
- C. Pinzari *Tensor categories and quantum groups*

List of invited talks : Banica, Capitaine, Enock, Isola, Cipriani, Landi, Morsella, Popa, Skandalis, Vasselli, Vergnioux, ...

Conference

Date : 30 August 3 September 2010
Location : Clermont-Ferrand

Quantum groups and interactions with

- Free probability
- Hopf-Galois
- Operator algebras
- Representation theory
- Tensor Categories

List of invited lecturers: Banica, Bruguieres, Caenepeel, Carnovale, Caspers, Colliins, Cuadra, Curran, De COmmer, Evans, Fima, Galindo, Guillot, Kassel, Launois, Lecouvey, Meir, Morrison, Mueger, Neshveyev, Neufang, Nikshych, Ostrik, Skalski, Voigt

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- Taste Camembert au lait cru ?

Web Links

Register now at

www.math.unicaen.fr/~aps/qsem