31èmes Rencontres arithmétiques de Caen
Abstracts

Courses

Huayi Chen: Arakelov geometry over adelic curves. In this series of lectures, I will explain a joint work with Atsushi Moriwaki on an Arakelov theory over a general countable field. Several classical constructions and results can be extended to this framework, including height of algebraic points and closed subvarieties, Hilbert-Samuel formula, positivity of adelic line bundles etc., which will be presented. I will also explain some points which are no longer valid in the general framework, which shows the necessity of applying slope method instead of Minkowski's geometry of numbers in the study of general adelic curves.

Fabien Pazuki: Northcott property and arithmetic applications. In the first part of this series of lectures, we will discuss the concept of Northcott numbers, and explain how to solve the related Northcott inverse problem. We will also describe applications of Northcott numbers, notably to undecidability questions, and more recently to provide examples of abelian varieties over infinite extensions with few small points, following Hultberg. In a second part, we will discuss questions related to the Northcott property for more exotic heights, in particular in the motivic setting, following Kato. We will see unconditional applications to families of values of Dedekind zeta functions.

We will conclude with a bundle of questions!

We base the exposition on joint work with Technau and Widmer, and on joint work with Pengo.

Research talks

Francesco Amoroso: Bounded height problems and applications. We shall report on some recent joint works with D. Masser and U. Zannier.

Let $\mathcal{C}$ be a curve defined over $\overline{\mathbb{Q}}$. In 1999, Bombieri, Masser and Zannier proved a result which may be rephrased as a toric analogue of Silverman's Specialization Theorem:

Let $\Gamma \subset \mathbb{G}_m(\mathcal{C})$ be a finitely generated subgroup of non zero rational functions on $\mathcal{C}$ which does not contain non trivial constant functions. Then the set of $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that the restriction of the specialization map $\sigma_P : \mathbb{G}_m(\mathcal{C}) \to \mathbb{G}_m(\overline{\mathbb{Q}}), x \mapsto x(P)$ to Γ is not injective is a set of bounded height.

Some years ago we prove, under some technical assumptions on Γ, the following generalisation:

Let V be an algebraic subvariety of $\mathbb{G}_m^r(\mathcal{C})$ and let $\sigma_P : \mathbb{G}_m^r(\mathcal{C}) \to \mathbb{G}_m^r(\overline{\mathbb{Q}})$ be the specialization map. Then the set of $P \in \mathcal{C}(\overline{\mathbb{Q}})$ such that for some $\mathbf{x} \in \Gamma^r \setminus V$ we have $\sigma_P(\mathbf{x}) \in \sigma_P(V)$ is a set of bounded height.

As a corollary we obtain a bounded height result for some degenerate unlikely intersections. Moreover, our specialisation result allows us to develop a new approach to treat families of norm form equations. We prove that, under suitable assumptions, all solutions of a norm form diophantine equation over an algebraic function field come from specialisation of functional equations. For instance for Thomas cubic equation we get:

All diophantine solutions $(t, x, y) \in \mathbb{Z}^3$ *of Thomas cubic equation*

$$X(X - A_1(T)Y)(X - A_2(T)Y) + Y^3 = 1$$

(with $A_1, A_2 \in \mathbb{Z}[T]$*,* $0 < \deg(A_1) < \deg(A_2)$*) for* $t \in \mathbb{N}$ *(effectively) large enough, are specialisations of a functional solution* (T, X, Y)*.*

A simple but significant example (already known by Beukers) of our specialization result is given by the family of equations $x^n + (1-x)^n = 1$ with n an integral parameter. In this simple case the height of the solutions is bounded by $\log(216)$. Very recently we make a start on the problem of generalising to rational exponents, which corresponds to the step from groups that are finitely generated to groups of finite rank. We discover some unexpected obstacles in principle. The proofs are partly based on our earlier work but there are also new considerations about successive minima over function fields.

Pascal Autissier: Distribution of the maximum of partial exponential sums. In analogy with character sums, I investigate in this talk the distribution of the maximum of partial exponential sums. More precisely, in a joint work with Bonolis and Lamzouri, we obtain precise uniform estimates for the distribution function of this maximum in a near optimal range, under a symmetry assumption. Proofs use analytic and probabilistic tools, as well as deep ingredients from algebraic geometry.

Fabrizio Barroero: Distinguished categories and the Zilber-Pink conjecture. The Zilber-Pink conjecture is a very general statement that implies many well-known results in diophantine geometry, e.g., Manin-Mumford, Mordell-Lang and André-Oort. I will report on recent joint work with Gabriel Dill in which we proved that the Zilber-Pink conjecture for a complex abelian variety A can be deduced from the same statement for its trace, i.e., the largest abelian subvariety of A that can be defined over the algebraic numbers. This gives some unconditional results, e.g., the conjecture for curves in complex abelian varieties (over the algebraic numbers this is due to Habegger and Pila) and the conjecture for arbitrary subvarieties of powers of elliptic curves that have transcendental j-invariant. While working on this project we realised that many definitions, statements and proofs were formal in nature and we came up with a categorical setting that contains most known examples and in which (weakly) special subvarieties can be defined and a Zilber-Pink statement can be formulated. We obtained some conditional as well as some unconditional results. If time allows I report on a further work in progress about distinguished categories and the Mordell-Lang conjecture.

Ana María Botero: Equivariant Chern Classes of Toric Vector Bundles over a DVR and Bruhat–Tits Buildings. We define equivariant Chern classes of a toric vector bundle over a toric scheme over a DVR. We describe them both as piecewise polynomial functions on the polyhedral complex associated to the toric scheme and in terms of the associated piecewise affine map from the polyhedral complex to the extended Bruhat–Tits building $\mathrm{GL}(r)$. We further discuss potential applications in the Arakelov theory of toric varieties. This is joint work with Kiumars Kaveh and Christopher Manon.

Laura Capuano: Betti maps, polynomial Pell equation and amost-Belyi maps. The classical Pell equation $A^2 - DB^2 = 1$ to be solved in integers A, B with $B \neq 0$ has a natural analogue in polynomials which goes back to Abel. In this context, the study of the locus of the polynomials D which fit in a Pell equation inside the space of polynomials of fixed even degree is related to the study of the Betti map of a particular section of the family of Jacobians of hyperelliptic curves and certain ramified covers from $\mathbb{P}^1 \to \mathbb{P}^1$ arising from the Pell equation and having heavy constrains on their ramification. In joint work with F. Barroero and U. Zannier we consider the "moduli space" of monic polynomials of fixed even degree $2d \geq 4$ and prove, among other things, that the locus of polynomials which fit in a Pell equation consists of a denumerable union of subvarieties of dimension at most $d + 1$.

Sara Checcoli: On a Galois property of fields generated by the torsion of an abelian variety. A field of algebraic numbers is said to have the Northcott property (N) if it contains only finitely many elements of bounded height for any given bound. This property, introduced by Bombieri and Zannier over twenty years ago, has been quite studied in recent decades. While its validity is trivial for number fields, it is generally a challenging task to determine it for infinite extensions of the rationals.

In the first part of my talk, I will survey the known non-trivial examples of fields that have the property (N). Specifically, let k be a number field, G be the multiplicative torus over k, and $k(G_{\text{tors}})$ be the smallest field containing k over which all torsion points of G are defined. It is not difficult to show, by adapting a result of Zannier and myself, that if L is a subfield of $k(G_{\text{tors}})$ containing k, such that L/k is Galois and $\mathrm{Gal}(L/k)$ is a group of exponent at most d, then L is contained in the compositum of all abelian extensions of k with degrees at most d. Consequently, L has the property (N), as proven by Bombieri and Zannier.

In the second part of my talk, I will show that a very similar result holds true when G is an abelian variety. This represents new joint work with Gabriel Dill.

Éric Gaudron: Diophantine approximation on spheres. Dirichlet's approximation theorem on spheres consists in approaching a vector located on the unit n-sphere by another vector on this sphere having rational coordinates. We present an adelic version of this theorem into the framework of rigid adelic spaces over some algebraic extensions of $\mathbb{Q}$. Our statements generalize and improve on earlier results by Kleinbock & Merrill (2015) and Moshchevitin (2017). The proofs rely on the quadratic Siegel's lemma obtained by the author & Gaël Rémond (2017).

Roberto Gualdi: On the typical height of the intersection of a projective planar line with its translate by a torsion point. The classical Bézout theorem affirms that the number of intersection points between two curves in the projective plane can be predicted from the geometric complexity of their defining polynomials, namely their degrees.

In a joint work with Martín Sombra we hint at a moral arithmetic version of this statement by the thorough study of a toy situation. More precisely, we show that the height of the projective point obtained as the intersection of the line $(x_0 + x_1 + x_2 = 0)$ with its translate by a torsion point is generically arbitrarily close to a value which can be predicted from the arithmetic complexity of the polynomial, and which turns out to be a quotient of special values of the Riemann zeta function.

The talk will focus on the precise statement of this result, on two different proofs for it (one of which is elementary!), on some numerical experiments, and on open questions.

Lars Kühne: The relative Bogomolov conjecture in products of families of elliptic curves. In the late nineties, Ullmo and Zhang proved the Bogomolov conjecture for subvarieties of abelian varieties, establishing that a geometrically irreducible subvariety of an abelian variety contains a Zariski dense subset of points with arbitrarily small Néron-Tate height if and only if it is an irreducible component of an algebraic subgroup. In the framework of unlikely intersections, generalizations of the Bogomolov conjecture to families of abelian varieties have been studied more recently. In my talk, I will discuss my proof of the conjecture in the case of fibered products of elliptic families. My result generalizes work of DeMarco and Mavraki and improves certain results of Manin-Mumford type proven by Masser and Zannier (or more recently: Gao and Habegger) to results of Bogomolov type.

Gaël Rémond: Petits points des variétés abéliennes. L'exposé présentera un travail en commun avec Éric Gaudron dans lequel nous donnons une majoration pour le nombre de points de petite hauteur de Néron-Tate d'une variété abélienne sur un corps de nombres. La borne, complètement explicite, améliore et généralise les résultats antérieurs connus, en particulier de Masser et David.

Martín Sombra: Arithmetic equidistribution beyond the quasi-canonical setting. For a projective variety X over a number field equipped with a metrized line bundle $\overline{L}$ satisfying some mild positivity assumptions, Zhang's inequality gives a lower bound for the essential minimum in terms of the corresponding height and degree of X. When this lower bound is attained, $\overline{L}$ is called *quasi-canonical*. In this situation, Yuan's equidistribution theorem dictates the asymptotic behavior of the Galois orbits of points with height converging to the essential minimum.

The notion of quasi-canonical metrized line bundle encompasses important cases like those corresponding to dynamical heights, including Néron-Tate heights on abelian varieties and canonical heights on toric varieties. However, it also leaves aside other interesting cases like canonical heights on semiabelian varieties and toric heights on toric varieties.

In this talk, I will present a result giving a weaker condition on $\overline{L}$ still ensuring the asymptotic equidistribution of these Galois orbits. This condition is verified for canonical heights on semiabelian varieties, allowing to recover a result of Chambert-Loir and Kühne's result in this direction. This result extends to the general situation the result by Burgos, Philippon, Rivera Letelier and the speaker for the toric case, and thus also allows to recover it.

Joint work with François Ballaÿ (Caen)